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INVERSE Γ—SEMIGROUP

N. K. SAHA AND A. SETH

Intro duction :

In [1] Sen and Saha defined a Γ—semigroup, regular Γ—semigroup and studied some properties of regular Γ—semigroup The aim of this paper is to introduce inverse Γ—semigroup. Along this way, we deduce necessary and sufficient condition for a Γ—semigroup to be inverse Γ—semigroup. We show that homomorphic image of an inverse Γ—semigroup an inverse Γ—semigroup.

1. Preliminaries.

Definition :

Let $M = \{ a, b, c, \ldots \}$ and $\Gamma = \{ x, y, z, \ldots \}$ be two non empty sets. M is called a Γ—semigroup if (i) $axb \in M$ and (ii) $(axb)\,yc = ax\,(byc)$ for all $a, b, c \in M$ and for all $x, y \in \Gamma$. A Γ—semigroup M will be denoted by $M(\Gamma)$. Since the associativity of the expression $(axb)\,yc = ax\,(byc)$ extends to all expressions of any length, henceforth we omit the bracket in the deduction of proofs of various statements.

Definition :

Let M be a Γ— semigroup. A non-empty subset B of M is said to be a Γ — sub semigroup of M if $B \Gamma B \subset B$.

Definition :

A right (left) ideal of a Γ — semigroup M is a non-empty subset I of M such that

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$I \Gamma M \subset I$ $(M \Gamma I \subset I)$. If I is both a right ideal and left ideal of M then we say that I is an ideal of M.

Definition :

Let M be a Γ – semigroup. An element $a \in M$ is said to be regular if $a \in a \Gamma M \Gamma a$ where $a\Gamma M\Gamma a = \{ axbya : b \in M \text{ and } x, y \in \Gamma \}$. A Γ – semigroup M is said to be regular if every element of M is regular.

Definition :

Let M be a Γ —semigroup. An element $e \in M$ is said to be an idempotent in M if there exists an $x \in \Gamma$ such that $exe = e$. In this case we say e is an x-idempotent.

Definition :

Let M be a Γ – semigroup and M_1 be a Γ_1—semigroup. A pair of mappings $f_1 : M \to M_1$ and $f_2 : \Gamma \to \Gamma_1$ is said to be a homomorphism from (M, Γ) to (M_1, Γ_1) if $(axb) f_1 = (af_1)(xf_2)(bf_1)$ for all $a, b \in M$ and $x \in \Gamma$ If f_1 and f_2 are both surjective then (f_1, f_2) is said to be a homomorphism from (M, Γ) onto (M_1, Γ_1).

Definition :

Let M be a Γ –semigroup. Let $a, b \in M$ and $x \in \Gamma$ If $axb = bxa$ we say that a, b are x–commutative. A Γ –semigroup M is said to be commutative if $axb = bxa$ for all $a, b \in M$ and for all $x \in \Gamma$.

Definition :

Let M be a Γ –semigroup and $a \in M$. Let $b \in M$ and $x, y \in \Gamma$. b is said to be an (x,y) inverse of a if $a = axbya$ and $b = byaxb$. In this case we shall write $b \in V_x^y(a)$. We may also write $b = a^{-1}_{x,\, y}$.

If $b \in V_x^y(a)$ then a and b are necessarily regular elements of M. On the other hand

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let $a = axcya$ where $c \in M$ and $x, y \in \Gamma$ be a regular element of M. Let $b = cyaxc$. Then $axbya = a$ and $byaxb = b$. Thus if $a = axcya$ be a regular element of M then b is an x, y) inverse of a where $b = cyaxc$.

2. Inverse Γ – semigroup

Definition :

A regular Γ – semigroup M is called an inverse Γ – semigroup if $|V_x^y(a)| = 1$, for all $a \in M$ and for all $x, y \in \Gamma$ whenever $V_x^y(a) \neq \phi$. That is every element a of M has a unique (x, y) inverse whenever (x, y) inverse of a exists.

Theorem (2, 1) :

Let M be a Γ—semigroup. M is an inverse Γ –semigroup if and only if (i) M is regular and (ii) if e and f be any two x-idempotents of M then $exf = fxe$, where $x \in \Gamma$.

Proof. Suppose M is an inverse Γ —semigroup. Then by definition M is regular. Next let and f be two x-idempotents of M. Let $a \in V_y^z(exf)$. Then $exfyazexf = exf$ . . . (1) and $azexfya = a$ . . (2) Now $fyazexfyaze = fyaze$ by (2). Therefore fyaze is an x-idempotent . . . (3). Also $exfxfyazexexf = exf$ by (1) and $fyazexexfxfyaze = fyaze$ by (3). Hence $exf \in V_x^x(fyaze$. But fyaze being x-idempotent belongs to $V_x^x(fyaze)$. Therefore $exf = fyaze$ since M is an inverse Γ – semigroup. Hence exf is an x-idempotent. Also, fxe is an x-idempotent of M. Indeed, if $b \in V_u^v(fxe)$, then $fxeubvfxe = fxe$ and $bvfxeub = b$. Then $eubvfxeubvf = eubvf$ Thus eubvf is an x—idempotent. Also, $fxexeubvfxfxe = fxe$ and $eubvfxfxexeubvf = eubvf$ Hence $fxe \in V_x^x(eubvf)$. But eubvf being x-idempotent belongs to $V_x^x(eubvf)$. Hence $fxe = eubvf$ Thus fxe is an x-idempotent of M. Now $exfxfxexexf = exfxexf = exf$ and $fxexexfxfxe = fxexfxe = fxe$. So, $fxe \in V_x^x(exf)$. But $exf \in V_x^x(exf)$. Therefore, $exf = fxe$.

Conversely let (i) and (ii) hold. We shall prove that M is an inverse Γ– semigroup. Let $b, c \in V_x^y(a)$ where $a \in M$. Then $axbya = a$ and $byaxb = b$ . . . (4). $axcya = a$ and $cyaxc = c$ . . . (5). Now each of axb and axc is y-idempotent of M and each of bya and cya is x-idempotent of M. Also $axbyaxc = axcyaxb$ using (ii). Therefore $axc = axb$. Similarly $byaxcya = cyaxbya$. Therefore $bya = cya$. Then $b = by(axb) = by(axc) = (bya)xc = (cya)xc = c$. Thus M is an inverse Γ — semigroup. This completes the proof.

We remember the following definitions :

Definition : Let A and B be two sets. Then any subset of $A \times B$ is called a binary relation from A to B.

Definition : A binary relation f from A to B is said to be an one-one partial transformation from A into B if the following conditions hold

(i) $(x,y) \in f$, $(x,y') \in f$ implies that $y = y'$

(ii) $(x',y) \in f$, $(x, y) \in f$ implies that $x = x'$

The set of all one-one partial transformations from A into B will be denoted by **J** (A, B).

Lemma (2.2) : **J** (A,B) is a Γ-semigroup, where $\Gamma =$ **J** (B,A).

Proof. Let f, g, h, . . . belong to **J** (A,B) and let $\alpha, \beta, \gamma, \ldots$ belong to **J** (B,A). We define $f\alpha g = \{ (a,b) \in A \times B :$ there exists $a_1 \in A$, $b_1 \in B$ such that $(a,b_1) \in f$, $(b_1,a_1) \in \alpha$, $(a_1, b) \in g \}$. Obviously, $f \alpha g$ is a binary relation from A to B. Let $(x, y) \in f\alpha g$ and $(x, y') \in f\alpha g$. Now $(x,y) \in f\alpha g$ implies there exist $a_1 \in A$, $b_1 \in B$ such that $(x,b_1) \in f$, $(b_1,a_1) \in \alpha$, $(a_1, y) \in g$ and $(x,y') \in f\alpha g$ implies there exist $a_2 \in A$, $b_2 \in B$ such that $(x,b_2) \in f$, $(b_2,a_2) \in \alpha$, $(a_2,y') \in g$. Now from $(x,b_1) \in f$ and $(x,b_2) \in f$ we get $b_1 = b_2$. Therefore $(b_1,a_1) \in \alpha$, $(b_2,a_2) \in \alpha$ impties that $(b_1, a_1) \in \alpha$, $(b_1, a_2) \in \alpha$. Hence $a_1 = a_2$. So, $(a_1,y) \in g$, $(a_2, y') \in g$ implies that $y = y'$. Thus $(x,y) \in f\alpha g$, $(x,y') \in f\alpha g$ implies that $y = y'$. Next let $(x,y) \in f\alpha g$ and $(x',y) \in f\alpha g$. Now $(x, y) \in f\alpha g$ implies there exist $a_1 \in A$, $b_1 \in B$ such that $(x,b_1) \in f$, $(b_1, a_1) \in \alpha$, $(a_1,y) \in g$ and $(x',y) \in f\alpha g$ implies there exist $a_2 \in A$, $b_2 \in B$ such that $(x',b_2) \in f$, $(b_2,a_2) \in \alpha$, $(a_2,y) \in g$. Then $(a_1,y) \in g$, $(a_2,y) \in g$ implies that $a_1 = a_2$.

So, $(b_1, a_1) \in \alpha$, $(b_2,a_2) \in \alpha$ implies that $b_1 = b_2$ and consequently $(x,b_1) \in f$. $(x',b_2) \in f$ implies $x = x'$. Hence $(x,y) \in f\alpha g$, $(x',y) \in f\alpha g$ implies that $x = x'$. Therefore $f\alpha g \in \mathbf{J} (A,B)$. Also, $(f\alpha g) \beta h = f\alpha (g\beta h)$. Indeed, $(a,b) \in (f\alpha g) \beta h$ if and only if there exist $a_1 \in A$, $b_1 \in B$ such that $\{ (a,b_1) \in f\alpha g \}$, $(b_1,a_1) \in \beta$, $(a_1,b) \in h$ if and only if there exist $a_2 \in A$, $b_2 \in B$ such that $\{(a, b_2) \in f, (b_2, a_2) \in \alpha, (a_2, b_1) \in g \}$, $(b_1, a_1) \in \beta$, $(a_1 ,b) \in h$ if and only if $(a, b_2) \in f$, $(b_2, a_2) \in \alpha$, $\{ (a_2, b_1) \in g, (b_1, a_1) \in \beta, (a_1, b) \in h \}$ if and only if $(a, b_2) \in f$, $(b_2, a_2) \in \alpha$, $(a_2, b) \in g\beta h$ if and only if $(a,b) \in f\alpha (g\beta h)$. Thus $(f\alpha g) \beta h=f\alpha (g\beta h)$. Hence $\mathbf{J} (A, B)$ is a Γ—semigroup.

Theorem (2 3) : $\mathbf{J} (A, B)$ is an inverse Γ-semigroup.

Proof. From Lemma (2. 2) we know that $\mathbf{J} (A, B)$ is a Γ-semigroup, where $\Gamma=\mathbf{J} (B,A)$. First we shall prove that $\mathbf{J} (A, B)$ is a regular Γ-semigroup.

Let $f \in \mathbf{J} (A, B)$. Then f : dom $f \to$ ran f is one-one. Let f^{-1} : ran $f \to$ dom f be such that $ff^{-1} = 1_{\text{dom } f}$ and $f^{-1}f = 1_{\text{ran } f}$, where $1_{\text{dom } f}$ denotes identity mapping on dom f. It is immediate that $f^{-1} \in \mathbf{J} (B,A)$. Now $f=ff^{-1}ff^{-1}f$ shows that $f \in f \,\Gamma\, \mathbf{J} (A,B) \,\Gamma\, f$. Thus $\mathbf{J}$ (A,B) is regular. Now to prove that J (A, B) is an inverse Γ-semigroup it is sufficient (from THEOREM 2.1) to show that any two α-idempotent of J (A, B) are α commutative where $\alpha \in \Gamma$. So we want to locate α-idempotents of J (A,B).

If an element $f \in \mathbf{J}$ (A,B) be α-idempotent then $f\alpha=1_X$, where $X =$ dom f and conversely if $f\alpha = 1_X$ where $X =$ dom f then f is α-idempotent. Now let f, h be any two α-idempotents in J (A B) We shall prove that $f\alpha h=h\alpha f$ .. (1).

Now $f\alpha=1_X$ where $X=$dom f and $h\alpha = 1_Y$ where $Y =$ dom h. Then

$$\text{dom} [f\alpha h] = \text{dom}[1_X h] =[X\cap Y]\, 1_X^{-1} = X\cap Y$$

$$\text{ran} [f\alpha h] = \text{ran} [1_X h] = [X\cap Y]\, h$$

$\mathrm{dom}\,[\,h\alpha f\,] = \mathrm{dom}\,[\,1_Y f\,] = [\,Y \cap X\,]\,1_Y^{-1} = Y \cap X = X \cap Y$

$\mathrm{ran}\,[h\alpha f] = \mathrm{ran}\,[1_Y f] = [Y \cap X]\,f = [X \cap Y]\,f$

We shall now show that $[X \cap Y]\,f = [X \cap Y]\,h$.
If possible let $[X \cap Y]\,f \neq [X \cap Y]\,h$. Then $[X \cap Y]\,f\alpha \neq [X \cap Y]\,h\alpha$.
Then $[X \cap Y]\,1_X \neq [X \cap Y]\,1_Y$ Then $X \cap Y \neq X \cap Y$, which is absurd. Hence $s\alpha h$ and $h\alpha f$ have same domain and range. Let $a \in X \cap Y$. Then $a = a$ implies $a1_Y = a1_X$ implies $ah\alpha = af\alpha$ implies $ah = af$ (since α is one-one) implies $[a1_X]\,h = [a1_Y]\,f$ implies $[af\alpha]\,h = [ah\alpha]\,f$ implies $af\alpha h = ah\alpha f$ So $f\alpha h = h\alpha f$. Thus (1) is proved. Hence J (A,B) is an inverse Γ—semigroup.

LEMMA (2.4) : Let M be an inverse Γ—semigroup. If e and f be two idempotents of M such that $exe = e$, $fyf = f$, $exf = eyf$, $fye = fxe$, then $exf = fxe$.

Proof. Let $a \in V_u^v(exf)$. Then $exfuavexf = exf$ and $avexfua = a$. Now $fuavexfyuave = fuave$. So, fuave is an x—idempodent. Also, $exfyfuavexexf = exf$ and $fuavexexfyuave = fuave$. Hence $exf \in V_x^y(fuave)$, . . . (1). Again $fuavexfuaveyfuave = fuav\,(exf)\,uav\;\;eyf)\,uave = fuav\,(\,exf\,)\,uav\,(\,exf\,)\,uave = fuavexfuave = fuave$.
Similarly, $fuaveyfuavexfuave = fuave$. So, $fuave \in V_x^y(fuave)$ . . . (2). From (1) and (2) $cxf = fuave$ (since M is an inverse Γ—semigroup). Therefore exf is an x—idempotent of M. Similarly fxe is an x—idempotent of M. Now $exfyfxexexf = exfxexf = exf$ and $fxexexfyfxe = fxexfxe = fxe$. Hence $fxe \in V_y^x(exf)$, But $exf \in V_y^x(exf)$. So, $exf = fxe$. This completes the proof.

LEMMA (2.5) : Let M be a regular Γ—semigroup and T be a Γ'—semigroup. Let (f,g) be a homomorphism from (M,Γ) onto (T,Γ'). Let e' be an x' —idempotent of T. Then $e'f^{-1}$ contains an idempotent of M

Proof. Let $a \in M$ be such that $af = e' = e'x'e'$, where $x' \in \Gamma'$. Let $x \in \Gamma$ be such that $xg = x'$. Now let us consider the element axa. As M is a regular Γ—semigroup there exist $c \in M$ and $y, z \in \Gamma$ such that $axayczaxa = axa$ and $czaxayc = c$. Now $aycza$ is an x—idempotent in M since $ayczaxaycza = aycza$

$$\begin{aligned} \text{Also, } (aycza)f &= (af)(yg)(cf)(zg)(af) = e'x'e'(yg)(cf)(zg)e'x'e' \\ &= (axaf)(yg)(cf)(zg)(axaf) = (axayczaxa)f = (axa)f \\ &= e'. \end{aligned}$$

Hence $e'f^{-1}$ contains an idempotent of M.

LEMMA (2 6) : Let M be a regular Γ—semigroup and let M' be a Γ'—semigroup. Let (f, g) be a homomorphism from (M, Γ) onto (M', Γ'). Then M' is a regular Γ'—semigroup.

Proof. Let $a' = af$ be an element of M' where $a \in M$. Since M is a regular Γ—semigroup there exist $b \in M$ and $x, y \in \Gamma$ such that $a = axbya$. Then $a' = af = (axbya)f = (af)(xg)(bf)(yg)(af) = a'(xg)(bf)(yg)a'$. Thus a' is regular. Hence M' is a regular Γ'—semigroup.

Theorem (2.7) : Let M be an inverse Γ—semigroup and M' be a Γ'—semigroup. Let (f,g) be a homomorphism from (M, Γ) onto (M', Γ'). Then M' is an inverse Γ'—semigroup. Moreover, in any homomorphism the (x,y) inverse of an element of M is mapped into the corresponding inverse of the image of the element

Proof. By LEMMA (2 6) M' is a regular Γ'—semigroup. We shall show that any two x'—idemotents of M' are x'—commutative, where $x' \in \Gamma'$. Let e_1', e_2' be two x'—idempotens of M'. Since (f,g) is onto homomorphism there exist e_1 and e_2 of M and $x \in \Gamma$ suce that e_1 and e_2 are x—idempotents of M and $e_i f = e_i'$, $xg = x'$, $i = 1,2$ [by LEMMA (2.5)]. As M is an inverse Γ—semigroup $e_1xe_2 = e_2xe_1$. Therefore $(e_1xe_2)f = (e_2xe_1)f$. So $e_1'x'e_2' = e_2'x'e_1'$. Hence M' is an inverse Γ'—semigroup. Moreover, for $a \in M$, $a^{-1}_{x,y}f = (af)^{-1}_{xg,\,yg}$. Indeed, $axa^{-1}_{x,y}ya = a$, $a^{-1}_{x,y}yaxa^{-1}_{x,y} = a^{-1}_{x,y}$. Therefore, $(af)(xg)(a^{-1}_{x,y}f)(yg)(af) = af$ and $(a^{-1}_{x,y}f)(yg)(af)(xg)(a^{-1}_{x,y}f) = (a^{-1}_{x,y}f)$. Thus $a^{-1}_{x,y}f = (of)^{-1}_{xg,\,yg}$. This completes the proof.

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REFERENCE

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COMPATIBLE TOPOLOGIES OF A GROUP AND THOSE OF ITS LATTICE OF SUBGROUPS—(II).

A. DAS GUPTA

1. It has been found [3] that the set of all compatible topologies of a group G and those of its lattice L (G) of subgroups form complete lattices T and L (T) respectiaely. Also, it has been proved that for a topology $t \in T$, we have a topology $t^* \in L(T$ and for the topology $t^* \in L(T)$ we have a topology $t' \in T$. Also it has been shown that $t' \leqslant t$. In this paper we have studied the condition under which t and t' are equal. We note that if $\cap U' = e$, $\forall\ U' \in \Sigma'$ where Σ' is the complete system of neighbourhoods of the identity e of the group G for tha topology t', then G with the topology t' is a Hausdorff space.

2. Let $t \in T$ and t' be that element of T which corresponds to $t^* \in L(T)$, where t^* is the corresponding element of t. [3]

Then we have the following :

Theorem—1 Let identity be the only common element of the complete system of neighbourhoods of identity for the topology t'. If t is compact, then $t = t'$ holds.

Proof : Consider the function $f : (G, t) \rightarrow (G, t')$ defined by $f(x) = x$, $\forall\ x \in G$ i. e. the identity function on G.

Now, the identity mapping f is algebrically an isomorphic mapping. Furthermore, the mapping f is continuous, since $t' \leqslant t$.

Hence, as t is compact and as $\cap\ U' = e$, $\forall\ U' \in \Sigma'$, where Σ' is the complete system of neighbourhoods of identity for the topology t', it follows that f is a homeomorphism.

Therefore, $t = t'$.

The converse of this theorem is not true i. e. $t' = t$ and $\bigcap U' = e, \forall\, U' \in \Sigma'$, does not imply that t is compact.

This can be shown from the following example.

Evample : Let G be the additive group of integers. Let p be a prime number, we denote by U_k the set of all integers, which are divisible by p^k.

We take for a complete system of neighbourhoods of zero, the totality Σ of all sets U_k, $k = 1, 2, 3. \ldots\ldots$ It can be easily verified that Σ becomes a complete system of neighbourhoods of identity for which G is a topological group.

Let us denote this topology by t.

Now, each U_k is algebrically a subgroup of G.

For, if $a, b \in U_k$ then $a - b \in U_k$.

From the topology t of T, we get the topology t^* in L (T) and from t^* we get t' in T.

Here obviously $t = t'$.

The topology t is not compact.

If posible, let t be compact.

Then $\{ U_k + n \}$ for $n = \pm 1, \pm 2, \pm 3, \ldots\ldots$ is an open cover of G.

From this open cover we can find a finite subcover of G.

That is, there exits integers, $n_1, n_2, \ldots\ldots n_m$ such that $G \subset \{U_k + n_1\} \cup \{U_k + n_2\} \cup \ldots\ldots \cup \{U_k + n_m\}$.

Let N be an integer such that $N - n_i$, $i = 1, 2 \ldots\ldots m$ is not divisible by p^k.

Let $N \in \{U_k + n_j\}$, where $1 \leqslant j \leqslant m$.

Then $a + n_j = N$ for some $a \in U_k$, or, $a = N - n_j \in U_k$. Contradiction as $N - n_j$ is not divisible by p^k and so, $a \notin U_k$.

3. We define a generalised topological group.

Definition :

A set G of elements is called a generalised topological group if

(1) G is an abstract group

(2) G is a topological space.

(3) The group operations in G are continuous in the topological space G.

Let G be a group and t be not the weakest compatible topology of G, so that G with the topology t is a generalised topological group.

Let t be a compact topology.

Then we can show that there exists a topology $\bar{t}$, for which the complete system of neighbourhoods of identity is $\bar{\Sigma} = \{ N, G \}$, where N is a normal subgroup and $\bar{t} \leqslant t$. For, from t we get $t' \mid t' \leqslant t$ [3].

Let H be any neighbourhood of identity belonging to t'.

Let $N = \bigcap_{x \in G} xHx^{-1}$. Here, $N \neq G$.

Since t is compact, there exists a neighbourhood V of $t \mid xVx^{-1} \subset H$ for all $x \in G$.

Hence, $N \subset x^{-1} Hx$, $\forall\, x \in G$. So, $V \subset N$.

Hence, N is an open normal subgroup of G.

We consider the set $\bar{\Sigma} = \{N, G\}$.

Then it can be shown that $\bar{\Sigma}$ satisfies all the five conditions :—

(1) Identity belongs to all the sets.

(2) The intersection of any two sets of the system $\bar{\Sigma}$ contains a third set of the system $\bar{\Sigma}$.

(3) For every set U of the system $\bar{\Sigma}$, there exists a set V of the same system, such that $VV^{-1} \subset U$.

(4) For every set U of the system $\bar{\Sigma}$ and an element $a \in U$, there exists a set V of the system $\bar{\Sigma}$, such that $Va \subset U$.

(5) If U is a set of the system $\bar{\Sigma}$ introduced above and 'a' an arbitrary element of the gronp G, then there exists a set V of the system $\bar{\Sigma}$, such that $a^{-1}Va \subset U$.

Hence $\bar{\Sigma}$ generates a compatible topology which we denote by $\bar{t}$. Also, it can be easily shown that $\bar{t} \leqslant t' \leqslant t$, where t and t′ have been introduced before.

Now, if G is a simple group, then the compact topology t is an atom in the lattice T.

For, let t_1 be any compatible topology such that $t_1 \leqslant t$.

Let Σ_1 be the complete system of neighbourhoods of identity of the topology t_1.

Then $\bigcap U = e. \forall U \in \Sigma_1$, since G is simple.

Hence G, with the topology t_1 is a Hausdorff space.

Also, t is compact Hence by theorem 1, $t_1 = t$ holds.

Therefore, t must be an atom.

Hence, we have $\bar{t} = t' = t$ holds, if G is simple.

Also, $\bar{t}$ is a discrete topology as $N = (e)$, the identity subgroup.

Hence, G with the topology t, is a compact discrete space. So, G is a finite group and T must be a two element lattice, as t is discrete and is an atom.

Therefore, we have the following theorem :

Theorem—2 :

If G be a simple group and $t \in T$ be a compact topology, then T is a two element lattice and G is finite.

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REFERENCES

[1] Birkhoff, G. Lattice Theory
Amer. Math. Soc.
Coll. Publ. 1963

[2] Chatterjee, B C. Abstract Algeqra
Das Gupta & Co. (Pvt) Ltd.

[3] Das Gupta, A. On compatible topologies of a group and those of lattice of subgroups.
(Communicated .

[4] Hussian, T. Introduction to Topological Groups.
W. B. Saunders Company.
Philadelphia and London 1966.

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ON REGULAR RINGS

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1. Regular rings were first introduced by J. V. Neumann and were subsequently studied by many prominent mathematicians. In this paper we have studied some properties of regular rings. Among other things we have proved that a regular ring is primitive iff it is prime. Further we have studied the nature of subdirectly irreducible regular rings

2. Here rings are all associative. Unless otherwise stated by an ideal we mean a both sided ideal. Let R be a ring and M be a right R-module. we shall say M is faithful if $Mr = (O)$, $r \in R$ implies $r = O$.

Let R be a ring and M be a right R-module. If $MR \neq (O)$ and if the only submodules of M are (O) and M, then M is called irreducible.

A ring will be called right primitive, if it has a faithful irreducible right R-module.

A ring R will be called prime, if for any to both sided ideals I and J, $IJ = (O)$ implies either $I = (O)$ or $J = (O)$.

A ring R (not necessarily with 1) will be called regular if for every a in R, $axa = a$ is solvable in R.

A ring R is said to be subdirectly irreducible if the intersection of all its non-null both sided ideals is different from the zero ideal.

An element a in a ring R is said to be right quasi-regular if there is an $a' \in R$ such that $a+a'+aa' = O$.

We say that a right ideal of R is right quasiregular if each of its elements is right quasiregular.

The radical of R written as J (R) is the set of all elements of R which annihilate all the irreducible R-modules.

If $J(R) = (O)$ then R is called semisimple. It can be proved $J(R)$ is a right quasiregular ideal.

Proposition 1. Any regular ring R is semisimple.

Proof : Let R be a regular ring and $a \in J(R)$. Now R being regular, $\exists\, x \in R$ such that $axa = a$. Put $ax = e$. Thus $e^2 = e$. Also $e \in J(R) \Rightarrow -e \in J(R)$. As $J(R)$ is a right quasiregular ideal, there exist $f \in J(R)$, such that $-e+f-ef = O \Rightarrow e = f - ef \Rightarrow e = e^2 = ef - ef = O$. Then $a = ea = O$. Thus R is semisimple

Theorem 1. Any subdirectly irreducible regular ring is primitive.

Proof : Let R be a subdirectly irreducible regular ring. As R is regular, it is semisimple. Then $\bigcap A(M) = (O)$, where M runs over all irreducible R-modules and $A(M)$ denotes its annihilator. It can be easily verified $A(M)$ is a both sided ideal.

As R is subdirectly irreducible, the intersection of all its nonnull ideal is not a zero ideal. Hence there exists $A(M_1)$ in R with $A(M_1) = (O)$ where M_1 is an irreducible R-module.

Thus R is primitive.

Corollary : Any subdircetly irreducible regular ring is prime.

Proof : It follows immediately from the theorem proved above, as every primitive ring is prime.

Converse is also true, and we give it in the following theorem.

Theorem 2. Any prime regular ring is subdirectly irreducible.

Proof : Let R be a prime regular ring. Let X denote the set of all non-null ideals in R

Consider the power set $P(X)$. $P(X)$ is a poset w.r.t usual set inclusion relation.

We define a property P on $P(X)$ as follows :

An element $A \in P(X)$ is said to have property P iff

$$\bigcap \{ I_\alpha, I_\alpha \in A \} \neq (O)$$

Evidently the minimal elements of $P(X)$ have the property P as they are of the form $\{I\}$, where I is a non-null ideal of R. Suppose the property P holds for every element, $Z < Y$ and

$Z \neq Y$ in $P(\mathcal{X})$. We shall prove that Y also has the property P. Let Z' denote the complement of Z in Y. Then $Z' < Y$ and $Z' \neq Y$ in $P(\mathcal{X})$.

By induction hypothesis $\cap \{ I_\alpha, I_\alpha \in Z \} \neq (O)$.

Denote it by S_1.

And $\cap \{ P_\alpha, P_\alpha \in Z' \} \neq (O)$ Denote it by S_2.

Then $\cap \{ T_\alpha, T_\alpha \in Y \} = S_1 \cap S_2$ (1)

Now $S_1 \neq (O)$, $S_2 \neq (O)$ implies $S_1 \cap S_2 \neq (O)$

If possible, let $S_1 \cap S_2 = (O)$.

Let $p \in S_1 \cap S_2$, then by regularity $p\,x\,p = p$ for some x in R. Now S_2 being both sided ideal $x\,p \in S_2 \Rightarrow p = p\,(x\,p) \in S_1S_2$, that is $S_1 \cap S_2 \subseteq S_1S_2$

Also $S_1S_2 \subseteq S_1 \cap S_2$. Thus $S_1 \cap S_2 = S_1S_2$.

Now R being prime, $S_1S_2 = (O) \Rightarrow S_1 = (O)$ or $S_2 = (O)$ contradiction shows that $S_1 \cap S_2 \neq (O)$.

Then from (1) we get Y has the property P By induction every element of $P(\mathcal{X})$ has property P.

As $\chi \in P(\chi)$, χ has the property P.

The intersection of all non-null ideals of R is not a zero ideal. Therefore R is subdirectly irreducible.

Combining Theorem 1 and Theorem 2 we have

Corollary : Any prime regular ring is primitive.

This answers a problem posed by I Kaplansky [7].

3. A ring in which every both sided principal ideal is generated by a central idempotent, is called a biregular ring.

Theorem 3. Any simple ring with 1, is a subdirectly irreducible biregular rlng.
Proof : The only both sided non-null ideal is the ring R. So R is subdirectly irreducible. Evidently R is biregular.

Theorem 4 : Any subdirectly irreducible biregular ring is a simple ring with 1.

Proof : Let R be a subdirectly irreducible biregular ring If possible let R be not simple.

Let $P = \cap \{ I_\alpha, I_\alpha \neq (O) \}$. Then $P \neq (O)$, as R is subdirectly irreducible.

Now $P \neq R$, for otherwise $P = R$ implies R is simple.

Thus P is a minimal ideal in R,

As $P \neq (O)$, there exists an element $a \in P/a \neq O \Rightarrow (a) \subseteq P$. As P is minimal $P = (a)$. As R is biregular, $P = (e)$, where e is a central idempotent. Let $(e)^*$ denote the annihilator of (e) Now $e \notin (e)^* \Rightarrow (e)^* \neq R$. If B is an ideal properly containing $(e)^*$, there exists an idempotent $f \neq O$ in the centre of R with $f \in B$, $f \notin (e)^*$.

But $(e) \subseteq (f)$. So $(e) \subseteq B$. Thus $(e) + (e)^* \subseteq B$ and so by Pierce decomposition $B = R$ and $(e)^*$ is maximal and hence $(e)^* \neq (O)$.

Now P being intersection of all non-null ideals $(e) \subseteq (e)^* \Rightarrow e = O$. We arrive at a contradiction. Thus R is simple.

Now as the centre is not a zero ring it is a field [2], and hence the centre contains an identity 1.

Thus $1 \in$ centre $R \subseteq R$.

We shall now prove that 1 is the identity of the whole ring R.

Let $a \in R$. Then $(a) = (e)$ for some central idempotent e. Thus $ae=a$.

$\therefore$, $a.1 = ae.1 = ae = a$

$\therefore$ 1 is the identity of R.

Thus R is a simple ring with 1.

4. A ring R is said to be strongly regular if for each element $a \in R$, there exists an $x \in R$ with $a^2x = a$.

It can be proved that strong regularity implies regularity and biregularity.

Proposition 5. Any skew field is a subdirectly irreducible strongly regular ring.

Proof is obvious.

Converse is also true.

Proposition 6. Any subdirectly irreducible strongly regular ring is a skew field.

Proof : Let R be any subdirectly irreducible strongly regular ring.

Then by Theorem 4, R is a simple ring with 1. Now as R is strongly regular, every idempotent is central, and as R is simple its centre is a field. Thus R contains only two idempotents viz O and 1. Let $O \neq a \in R$. Then $\exists\, x \in R \mid a x a = a \Rightarrow$ ax is an idempotent $a \neq O \Rightarrow ax \neq O$ thus $ax = 1$. Thus R is a skew field.

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References.

[1.] A. Kurosh—Lectures in General algebras. London Permagon Press, 1965.

[2.] I.N. Herstein—Non commutative rings The Carus Mathematical Monographs Number Fifteen.

[3.] L. A Ckophyakob—Degokuhgobor cmpykmypoc c gopolhehyauu u pezylyaphbe kolbua Zocygapcmbehhoe uzgamelb cmbo
fuzuko-Mamemamucheckou lumepamyoe, Mockba 1961

[4.] J.V. Neumann—Continuous Geometry. Princeton University Press 1960.

[5.] Mathew C. Waddel—Properties of regular ring. Duke Mat J. 19 (1952) 623-627.

[6.] Kwangil Koh—On Von Neumann regular ring. Canad Math Bull. Vol. 17(2) 1974.

[7.] I. Kaplansky—Canadian Mathematical Bulletin Vol. 14 No. 4 1971

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ON THE MAXIMUM TERMS, ORDERS AND TYPES OF THE DERIVATIVES OF AN ENTIRE FUNCTION IN SEVERAL COMPLEX VARIABLES.

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Abstract : Let F be the family of all entire functions in the complex n-space C^n . For f, g ϵ F, the Hadamard product $f * g$ is defined. Certain inequalities involving maximum modulii of the derivatives of the Hadamard product and the Hadamard product of the derivatives of f and g have been obtained. A few relations involving maximum term and the corresponding rank of the derivatives of the above product have also been obtained. Two kinds of orders and types have been considered and a few results involving them have been obtained.

1. **Notations :** We denote complex and real n-space by C^n and R^n respectively and the set of non-negative integers by I, so that I^n denotes the Cartesian product of n-copies of I. We indicate the points $(z_1, \ldots , z_n)$, $(r_1, \ldots, r_n)$, $(m_1, \ldots , m_n)$ etc of C^n or R^n by their corresponding unsuffixed symbols z, r, m, etc. For z, w ϵ C^n and $\alpha \epsilon C$ we define z = w iff

$z_i = w_i, i = i, \ldots, n$

$\alpha z = \alpha z_1, \ldots \alpha z_n)$

$z + w = (z_1 + w_1, \ldots, z_n + w_n)$;

$| z | = \{ | z_1 |^2 + \ldots + | z_n |^2\}^{\frac{1}{2}}$

The positive hyper octant R_+^n is the set $R^n = \{ x : x \epsilon R^n , x_i \geqslant o, i=1, \ldots, n\}$. For $k \epsilon R_+^n$ we set $\| k \| = k_1 + \ldots + k_n$ and for $m \epsilon I^n$, $m ! = m_1 ! \ldots m_n !$.

For any $p \epsilon I$, $\bar{p}$ will stand for the n-tuple $(p, \ldots, p)$. Also for $z \epsilon C^n$, $k \epsilon R_+^n$, we shall

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write $z^k = z_1^{k_1} \dots z_n^{k_n}$ ($z_i^0 = 1$ even if $z_i = 0$).

For any x, y $\in R^n$, we say that

(1) $x \leqslant y$ iff $x_i \leqslant y_i$, $i = 1, \dots, n$

(2) $x < y$ iff $x \leqslant y$ but $x \neq y$

(3) $x << y$ iff $x_i < y_i$, $i = 1, \dots, n$

For an entire function f with domain C^n, $f^{(k)}$ will denote the function

$$\frac{\partial^{\| k \|} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}, \quad k \in I^n.$$

We write for any non-empty complete n-circular domain D [for definition vide 1 § 3.3] with centre at $\bar{O} = (O, \dots, O)$ in C^n. $| D | = \{ r : | z_i | = r_i, z \in D \}$ and $D^+ = \{ r : r \in | D |, \text{ no } r_i = O \}$

For any $r = (r_1, \dots, r_n) \in R^n$ we write $r = \prod_{i=1}^{n} r_i$

2. Let F be the family of all entire functions in C^n represented by a multiple power series of the form

$$f(z) = \sum_{\| m \| = o}^{\infty} a_m z^m \tag{2.1}$$

For f, g $\in$ F we define Hadamard product $f * g$ by

$$(f * g)(z) = \sum_{\| m \| = o}^{\infty} a_m b_m z^m, \quad \text{where} \tag{2.2}$$

$$g(z) = \sum_{\| m \| = o}^{\infty} b_m z^m \tag{2.3}$$

we see

$$f^{(k)}(z) = \sum_{\| m \| = o}^{\infty} \left[\left[\prod_{i=1}^{n} \left\{ \prod_{j=o}^{k_i - 1} (m_i - j) \right\} \right] \right] a_m z^{m-k} \tag{2.4}$$

Evidently $f * g$ belongs to F.

Corresponding to any $f \in F$ we define the maximum modulus $M(r, f)$ on R_+^n by $M(r, f) = \max_{|z_i| = r_i} |f(z)|$. Throughout this section $M(r, k)$ and $M^*(r, k)$ will respectively denote the maximum modulus of the functions $(f * g)^{(k)}$ and $f^{(k)} * g^{(k)}$ on $|z_i| = r_i$.

Evidently $M(r, \bar{O}) = M^*(r, \bar{O})$ for any $r \in R_+^n$ We also see that

$$(2.5) \qquad M(r, k) \leqslant k! \frac{M(R, \bar{O})}{(R-r)^k} \quad \text{for } \bar{O} \leqslant r << R \text{ and } k \in I^n.$$

Theorem 1. For $f, g \in F$, as defined by (2.1) and (2.3)

$$(2.6) \qquad M^*(r, k) \leqslant \frac{k!\, R^k M(r, k)}{(R-r)^k} \leqslant \frac{(k!)^2 R^k M(R', \bar{O})}{(R-r)^k (R'-R)^k}$$

where $\bar{O} < r << R << R'$ and $k \in I^n$

Proof. We have

$$\frac{\partial^{\|k\|}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \left\{ z^k (f(z) * g(z))^k \right\} = f^{(k)} * g^{(k)}$$

Then, for any z such that $|z_i| = r_i$ $i = 1 \ldots n$ by Cauchy's Integral Formula

$$f^{(k)}(z) * g^{(k)}(z) = \frac{k!}{(2\pi i)^n} \int_C \frac{t^k (f(t) * g(t))^{(k)}}{(t-z)^{k+1}} dt_1 \ldots dt_n$$

where $C = C_1 \times \ldots \times C_n$, $C_i = |t_i - z_i| = R_i - r_i$, $i = 1. \ldots n$.

Therefore

$$M^*(r, k) \leqslant \frac{k!\, R^k M(r, k)}{(R-r)^k}$$

The other part immediately follows from (2 5)

3. Now corresponding to $f \in F$, we define the function maximum $\mu_f(r)$ and central indices $\nu_j(r)$ of $\mu_t(r)$ $j = 1, \ldots, n$ on R^n by

$$\mu_f(r) = \underset{m \in I^n}{\mathrm{Max}} \{ | a_m | r^m \}$$

$$\nu_i(r) = \mathrm{Max}\{ m_i : | a_m | r^m = \mu_f(r) \ i = 1, \ldots, n \}$$

We call $\nu = (\nu_1, \ldots, \nu_n)$ as the rank of the maximum term $\mu_f(r)$. We shall throughout denote by $\mu(r, k)$ and $\mu^*(r, k)$ the maximum terms of $((f * g)(z))^{k)}$ and $f^{(k)}(z) * g^{(k)}(z)$. Then

$$(3.1) \qquad \mu(r, k) = \underset{m \in I^n}{\max} \left[\prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (m_i - j) \right\} | a_m b_m | r^{m-k} \right]$$

and

$$(3.2) \qquad \mu^*(r, k) = \underset{m \in I^n}{\max} \left[\prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (m_i - j)^2 \right\} | a_m b_m | r^{m-k} \right]$$

Also let

$\nu_j = \nu_j(r, k)$ and $\nu_j^* = \nu_j^*(r, k)$ $i=1, \ldots, n$

be the central indices of $\mu(r, k)$ and $\mu^*(r, k)$ respectively and

$\nu = \nu(r, k) = (\nu_1, \ldots, \nu_n)$ and $\nu_1^* = \nu_n^*(r, k) = (\nu_1^*, \ldots, \nu_n^*)$ be their respective ranks.

In this section we obtain a few relations between $\mu^*(r, k)$ and $\mu(r,k)$ which give us more information about the class of entire functions defined by (2.1). For any $f \in F$, let D_1 be the set of discontinuities of ν in $| C^n |$ where $\nu = (\nu_1 \cdots, \nu_n)$ is the rank of $\mu_f(r)$. Also let S dcnote the set of all $r \in | C^n |$ at which $\mu_f(r)$ is attained by more than one term of the series

$$(3.3) \qquad \sum_{\| m \| = 0}^{\infty} a_m r^m$$

[3, J. Gopal Krishna] had shown that D_1 and S are identical.

Hence for $r \in | C^n | - D_1$, $\mu_f(r)$ is attained by only one term of the series (3.3) and the position of that term is $\nu = (\nu_1, \ldots, \nu_n)$.

Theoram 2. For $r \in | C^n | - D_1 \cup D_1^*$ and $k \in I^n$

$$(3.4)\qquad \prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}\nu_i-j)\right\}\leqslant\frac{\mu^*(r,k)}{\mu(r,k)}\leqslant\prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-j}(\nu_i*-j)\right\}$$

where D_1 and D_1^* denote the set of discontinuities of ν and $\nu*$ in $|\,C^n_{+}\,|$, ν and $\nu*$ are he ranks of $\mu(r,k)$ and $\mu^*(r,k)$

Proof. From (3.1)

$$(3.5)\qquad \mu(r,k)\geqslant\frac{\mu^*(r,k)}{\prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}(\nu_{i*}-j)\right\}}$$

Also from (3.2)

$$(3.6)\qquad \mu^*(r,k)\geqslant\prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}(\nu_i-j)^2\right\}|\,a_\nu b_\nu\,|\,r^{\nu-k}$$

$$=\mu(r,k)\prod_{i=1}^{n}\left\{\prod_{j=0}^{k_i-1}(\nu_i-j)\right\}.$$

Hence the the theorem follows from (3 5) and (3.6).

Theorem 3. If $\nu\,(r,k)$ and $\nu\,(r,k+\bar{I})$ be the ranks of $\mu(r,k)$ and $\mu(r,k+\bar{I})$. Then

$$(3\ 7)\qquad \prod_{i=1}^{n}\left\{\nu_i(r,k)-k_i\right\}\leqslant\frac{\mu(r,k+\bar{I})}{\mu(r,k)}\;\pi r$$

$$\leqslant\prod_{i=1}^{n}\left\{\nu_i(r,k+\bar{I})-k_i\right\}$$

for $r\in|\,C^n\,|-D_1\cup D_2$, $k\in I^n$, where D_1 and D_2 are the set of discontinuities of $\nu\,(r,k)$ and $\nu(r,k+\bar{I})$ in $|\,C^n\,|$.

Proof. From (3.1), we have

$$\mu(r,k+\bar{I})=\prod_{i=1}^{n}\left[\nu_i(r,k+\bar{I})\ \ldots,\ \left\{\nu_i(r,k+\bar{I})-k_i\right\}\right]$$

$$|\,a_{\nu(r,k+\bar{I})}b_{\nu(r,k+\bar{I})}\,|\,r^{\nu(r,k+\bar{I})-k-\bar{I}}$$

Therefore,

$$\mu(r,k+\bar{1}) \leqslant \mu(r,k) \frac{\prod_{l=1}^{n} \left\{ \nu_i (r,k+\bar{1}) - k_i \right\}}{\prod_{i=1}^{n} r_i}$$

i. e.

$$\frac{\mu(r,k+\bar{1}}{\mu(r,k)} \pi r \leqslant \prod_{i=1}^{n} \left\{ \nu_i (r,k+\bar{1}) - k_i \right\} \tag{3.8}$$

Again

$$\mu(r,k+\bar{1}) \geqslant \prod_{i=1}^{n} [\ \nu_i \ \ r,k) \quad \left\{ \nu_i (r,k) - k_i \right\}]$$

$$| a_{\nu}(r,k) b_{\nu} (r,k) | \ r^{\dot{\nu} (r,k) - k - \bar{1}}$$

$$= \prod_{i=1}^{n} \left\{ \nu(r,k) - k_i \right\} \frac{\mu(r,k)}{\pi r} \tag{3.9}$$

(3 8) together with (3.9) proves (3.7)

Corollaries.

1) $\prod_{l=1}^{n} \nu_i (r,\bar{0}) \leqslant \prod_{i=1}^{n} \nu_i (r,\bar{1}) \leqslant \cdots$

2) $\frac{\mu(r,\bar{1})}{\mu(r,\bar{0})} \leqslant \frac{\mu(r,2)}{\mu(r,\bar{1})} \leqslant \cdots$

3) Putting $k=\bar{0}, \ldots \overline{p\text{-}1}$ successively in (3.7) and from above we get

$$\prod_{i=1}^{n} \nu_i (r,\bar{0}) \leqslant \left\{ \frac{\mu(r,\bar{p})}{\mu(r,\bar{0})} \right\}^{1/p} \leqslant \prod_{i=1}^{n} \nu_i (r,\bar{p})$$

If we do not delete the set of discontinuities of ν and ν^*, the above theorems take the following forms whose proofs follows in the same line as those of the above.

Theoram 2′ For any $r \in | C^n |$, $k \in I^n$

$$\prod_{i=1}^{n}\left\{\prod_{j=o}^{k_i-1}(p_i-j)\right\} \leqslant \frac{\mu^*(r,k)}{\mu(r,k)} \leqslant \prod_{i=1}^{n}\left\{\prod_{j=o}^{k_i\;\;1}(\nu_i*-j)\right\}$$

where $p=(p_1, \quad ,p_n)$ is a position of occurrence of $\mu(r,k)$ and $\nu^*(r,k)=(\nu_1^*,\cdots,\nu_n^*)$ is the rank of $\mu^*(r,k)$.

Theorem 3′ If $q=(q_1, \quad ,q_n)$ is a position of occurrence of $\mu(r,k)$ and $\nu=(\nu_1,\cdots\nu_n)$ is the rank of $\mu(r,k+\bar{I})$, then, for any $r \in | C^n |$

$$\prod_{i=1}^{n}\left\{q_i(r,k)-k_i\right\} \leqslant \frac{\mu(r,k+\bar{I})}{\mu(r,k)}\,\pi r$$
$$\leqslant \prod_{i=1}^{n}\left\{\nu_i(r,k+\bar{I})-k_i\right\}$$

4. In this section we shall consider Gol'dberg order and Gol'dberg type of an entire function in C^n. Let $D \subset C^n$ be an arbitrary bounded complete n-circular domain with centre at the origin of coordinates. Then for the entire function f, we define

$M_{f,\,D}(r) = \sup_{z \in D_r} | f(z) |$, $r(> O) \in R$, where the point $z \in D_r$ iff the point $\left(\frac{z_1}{r}, \ldots\ldots, \frac{z_n}{r}\right) \in D$. The Gol'dberg order ρ_D and Gol'dberg type σ_D (briefly G-order and G-type) of f w. r. t the domain D are defined by the formulas [1, Fuks P. 339]

$$P_D = \limsup_{r \to \infty} \frac{\log \log M_{f,\,D}(r)}{\log r} \qquad \sigma_D = \limsup_{r \to \infty} \frac{\log M_{f,\,D}(r)}{r^D}$$

It turns out that the G-order ρ_D does not depend on the choice of the domain D while the G-type σ_D does [1, P.339]. It is also known [1, P.339] that for the entire function f

(4.1) $$P_D = P = \limsup_{\| m \| \to \infty} \frac{\| m \| \log \| m \|}{-\log | a_m |} \text{ and}$$

(4.2) $$(e\rho\sigma_D)^{1/\rho} = \limsup_{\|m\|\to\infty} \left\{ \|m\|^{1/\rho} \left(|a_m|\, d_m(D) \right)^{\frac{1}{\|m\|}} \right\}$$

where $$d_m(D) = \sup_{z\in D} \left\{ |z_1|^{m_1} \cdots |z_n|^{m_n} \right\}$$

Theorem 4 If f is an entire function with G-order ρ ($0 < \rho < \infty$), then $f^{(k)}$ ($k \in I^n$) is also of G-order ρ. Moreover, for a bounded complete n-circular domain D, the entire function f and $f^{(k)}$ have he same G-type σ_D.

Proof. Let $f(z) = \sum_{\|m\|=0}^{\infty} a_m z^m$ be of G-order ($0 < \rho < \infty$). Then, from (2.4)

$$f^{(k)}(z) = \sum_{\|m\|=0}^{\infty} \left[\prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (m_i - j) \right\} \right] a_m z^{m-k}$$

Now $$\liminf_{\|m\|\to\infty} \frac{-\log\left[|a_m| \prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (m_i-j) \right\} \right]}{\|m\| \log \|m\|}$$

$$= \liminf_{\|m\|\to\infty} \left[\frac{-\log |a_m|}{\|m\| \log \|m\|} - \frac{\log \prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (m_i-j) \right\}}{m \log m} \right]$$

$$= \liminf_{\|m\|\to\infty} \frac{-\log |a_m|}{\|m\| \log \|m\|} = \frac{1}{\rho} \quad \text{by (4.1)}$$

Again $$\limsup_{\|m\|\to\infty} \left[\|m\|^{1/\rho} \left\{ |a_m|\, d_m(D) \prod_{i=1}^{n} \left(\prod_{j=0}^{k_i-1} (m_i-j) \right) \right\}^{\frac{1}{\|m\|}} \right]$$

$$= \limsup_{\|m\|\to\infty} \left[\|m\|^{1/\rho} \left[|a_m|\, d_m(D) \right]^{\frac{1}{\|m\|}} \right]$$

which proves the Theorem,

Remark. In a similar way we can prove the following :

For f, g ϵ F, if f $*$ g is of G-order ρ ($O < \rho < \infty$) and G-type σ_D corresponping to a bounded complete n-circular domain D then $f^{(k)} * g^{(k)}$ is also of G-order ρ and G-type σ_D.

Theroem 5. Let f, g ϵ F be of G-osders ρ_1 and ρ_2 respectively, then f $*$ g ϵ F and satisfy $1/\rho \geqslant 1/\rho_1 + 1/\rho_2$ where ρ is the G-order of f $*$ g.

Proof. That f $*$ g ϵ F is evident Now,

$$1/\rho = \liminf_{\| m \| \to \infty} \frac{-\log | a_m \; b_m -}{\| m \| \log \| m \|}$$

$$(4.3) \qquad \geqslant 1/\rho_1 + 1/\rho_2 \qquad \text{by } (4\,1)$$

Corollary. If f, g ϵ F be of G-orders ρ_1 and ρ_2 respectively then $f^{(k)} * g^{(k)}$ ϵ F and be of G-order ρ satisfying (4.3)

5. Let f be an entire function and M (r) be its maximum modulus. Let B_f denote the set (may be empty) of all points $\alpha \epsilon R^n_+$ such that

$$\log M(r) < r_1^{\alpha_1} \ldots + r_n^{\alpha_n} \text{ for } \| r \| \to \infty .$$

The boundary ∂B_f of the set B_f is called the order of f and any point $\rho \epsilon \partial B_f$ is called an order point. We say that f is of finite or infinite order according as B_f is non empty or empty. Evidently, for any $\rho \epsilon \partial B_f$, $\rho \geqslant \bar{O}$ An order point ρ is said to be positive if $\rho >> \bar{O}$.

Next, let $\rho \epsilon \partial B_f$ ($\rho >> \bar{O}$) and $T_f = T_f(\rho)$ denote the set (may be emyty) of all points $\beta \epsilon R^n_+$ such that $\log M(r) < \beta_1 r_1^{\rho_1} + \ldots + \beta_n r_n^{\rho_n}$ for $\| r \| \to \infty$.

The boundary ∂T_f of the set T_f is called the type of f corresponding to the order point ρ. A point $\sigma \epsilon \partial T_f$ is called a type point of f. A type point σ is called positive if $\sigma >> \bar{O}$.

We say that f is of finite or infinite type according as T_f is non-empty or empty.

Theorem 6. The entire functions f and $f^{(k)}$ ($k \in I^n$) have the same set of positive order points. Moreover for a positive order point they have the same set of positive type points.

Proof. Let f be as in (2.1) and let ρ be an order point of f. It is known [P. 137 Ronkin] that ρ ($>> \bar{O}$) is an order point of f iff

$$(5.1)\qquad \limsup_{\|m\| \to \infty} \frac{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n}{-\log |a_m|} = 1$$

Now,

$$\limsup_{\|m\| \to \infty} \frac{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n}{-\log \left[\prod_{i=1}^{n} \left(\prod_{j=o}^{k_i-1} (m_i - j) \right) |a_m| \right]}$$

$$= 1 \Big/ \liminf_{\|m\| \to \infty} \frac{-\log \left[\prod_{i=1}^{n} \left(\prod_{j=o}^{k_i-1} (m_i - j) \right) |a_m| \right]}{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n}$$

$$= 1 \Big/ \liminf_{\|m\| \to \infty} \frac{-\log |a_m|}{\frac{m_1}{\rho_1}\log m_1 + \dots + \frac{m_n}{\rho_n}\log m_n} = 1 \qquad \text{by } (5.1)$$

By reversing the step we get the converse part which settles the first part.

It is known [P.139, Ronkin] that $\sigma(>>\bar{O})$ is a type point of f for a positive order point ρ iff

$$(5.2)\qquad \limsup_{\|m\| \to \infty} \left\{ |a_m| \prod_{i=1}^{n} \left(\frac{m_i}{e\sigma_i\, \rho_i} \right)^{m_i/\rho_i} \right\}^{\frac{1}{\|m\|}} = 1$$

Let $\sigma(>>\bar{O})$ be a type point of f the positive order point ρ so (5.2) holds.

Now,

$$\limsup_{\|m\|\to\infty} \left[|a_m| \prod_{i=1}^{n} \left(\frac{m_i}{e\sigma_i \rho_i}\right)^{m_i/\rho_i} \prod_{i=1}^{n} \left\{ \prod_{j=0}^{k_i-1} (m_i - j) \right\} \right]^{\frac{1}{\|m\|}}$$

$$(5.2) \qquad = \limsup_{\|m\|\to\infty} \left[|a_m| \prod_{i=1}^{n} \left(\frac{m_i}{e\sigma_i \rho_i}\right)^{m_i/\rho_i} \right]^{\frac{1}{\|m\|}} = 1 \text{ by}$$

Hence σ is a type point of $f^{(k)}$ for the order point ρ. Reversing we get converse part. This completes the proof.

Theorem 7. If $f, g \in F$ then $f * g$ and $f^{(k)} * g^{(k)}$ ($k \in I^n$) have the same set of positive order points. Further, for a positive order point they have the same set of positive type point.

Proof. The proof is exactly similar to that of Theorem 6.

Remark. We observe that while our order points and type points are subsets in R^n_+ the G-order and G-type of an entire function are simply non-negative real numbers.

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REFERENCES

[1] Fuks, B. A. : Introduction to the theory of Analytic functions of several complex variables-Amer. Math. Soc. (1963).

[2] Gol'dberg A. A. : Elementary remarks on the formulas defining order and type of functions of several variables, Akad. Nank Armjan, SSR Dokl. 29 (1959).

[3] Gopal Krishna, J. : Maximum term of a power series in one and several complex variables - Pacific Jour. of Math. Vol. 29 No. 3 (1969) pp. 609-622.

[4] Ronkin, L. I. : Introduction to the theory of entire functions of several variables - Amer. Math. Soc. (1974).

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ON PAIRWISE D-SPACES

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ABSTRACT In this paper, pairwise D-bitopological spaces have been introduced and studied.

KEY WORDS AND PHARASES Pairwise D-Space & D-subset, pairwise semi-open, pairwise semi-continuous, pairwise regularly open, pair-wise D-separation, pairwise D-connected,

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1. **INTRODUCTION** Norman Levine [2] defined and studied D-topologies He called a topological space (X, T) a D-space if T is a D-topology on X, that is, if every nonempty open set of (X, T) is dense in X. It is the aim of this paper to introduce this concept in bitopological spaces. In Section 2, the notion of pairwise D-spaces is introduced and it is shown that a pairwise D-space is not just a pair of D-spaces. Some properties of such pairwise D-spaces have been investigated and certain theorems of [2] have been generalized. In Section 3, we define pairwise D-connected bitopological space and the relation of such concept with that of connected bitopological space is studied.

In course of the exposition of the paper, we shall require some definitions which we state below. By (X, T_1, T_2) or simply by X we shall always mean a bitopological space with two topologies T_1 and T_2, $A^{t}1$, $A^{t}2$ shall mean T_1-interior and T_2-interior respectively of a subset A of X, whereas by $\bar{A}_{T_1}$ and $\bar{A}_{T_2}$ will be denoted respectively the T_1-closure and T_2 closure of A in (X, T_1, T_2,). A bitopological space (X, T_1, T_2) is said to satisfy pairwise T_1-axiom [4] if for any two distinct points x, y of X, there exist T_1-open set U and T_2-open set V such that

$x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. (X, T_1, T_2) is called pairwise Hausdorff [1] if for any pair of distinct points x, y of X, there are T_1-open neighbourhood V of x and T_2-open neighbourhood W of y such that $V \cap W = \Phi$, A subset A of X is called T_i semiopen with respect to T_j [5], if there is a T_i open set V such that $V \subset A \subset \bar{V}^{T_j}$, where i, j=1, 2 and $i \neq j$. A is called pairwise semiopen if it is T_1 semiopen w.r.t. T_2 and T_2 semiopen w.r.t T_1. A function f from (X, T_1, T_2) to a bitopological space (Y, P_1, P_2) will be called T_1 P_1—semi continuous w.r.t. T_2 [5], if for each P_1-open set A, $f^{-1}(A)$ is T_1 semiopen w.r.t. T_2. Similarly T_2 P_2-semi continuity of f w r.t. T_1 is defined. f is called pairwise semi continuous if f is T_1P_1-semicontinuous w.r.t. T_2 and T_2P_2-semi continuous w r.t T_1. A subset A in (X, T_1, T_2) is called T_1 regularly open w.r.t. T_2 [5], if and only if $A = (\bar{A}^{T_2})^{i_1}$. Similar goes the definition of T_2 regularly open set w r.t. T_1.

2. PAIRWISE D-SPACES AND THEIR PROPERTIES

DEFINITION 2. 1 Let (X, T_1, T_2) be a bitopological space. We easily note that every nonempty T_1 open set of X is T_2-dense in X if and only if every nonempty T_2 open set is T_1 dense in X.

We define (X, T_1, T_2) to be a pairwise D-space if and only if every nonempty T_1 open set is T_2 dense in X (or equivalently, if and only if every nonempty T_2 open set is T_1 dense in X).

It follows immediately that

THEOREM 2.2 $(X. T_1, T_2)$ is pairwise D-space if and only is every nonempty T_1 open set has nonempty intersection with every nonempty T_2 open set

REMARK 2.3 It follows from definition that a pairwise D-space can never be pairwise Hausdorff, though it may satisfy pairwise T_1-axiom as is seen from

EXAMPLE 2.4 Let X be an uncountable set and T_1 be the cofinite topology and T_2 - the co-countable topology on X. Then (X, T_1, T_2) is a pairwise D-space and satisfies T_1-axiom.

REMARK 2.5 A bitopological space (X, T_1, T_2) is pairwise D-space does not imply that (X, T_1) or (X, T_2) is a D-space. Also, (X, T_1, T_2) may not be a pairwise D-space even if (X, T_1) and (X, T_2) are D-spaces. This is seen in the next example.

EXAMPLE 2.6 Let $X = \{a, b, c, d\}$, $T_1 = \{X, \phi, \{a\}\}$ and $T_2 = \{X, \phi, \{b\}\}$. Then (X, T_1) and (X, T_2) are D-spaces but (X, T_1, T_2) is not a pairwise D-space.

Again, let $P_1 = \{X, \phi, \{a, b\}, \{c, d\}\}$ and $P_2 = \{X, \phi, \{a, c\}, \{b, d\}\}$. Then (X, P_1, P_2) is a pairwise D-space whereas neither (X, P_1) nor (X, P_2) is a D-space.

THEOREM 2.7 Every subspace of a pairwise D-space is also a pairwise D-space.

THEOREM 2.8 (X, T_1, T_2) is pairwise D-space if and only if X contains no proper pairwise T_1 regularly open set w.r.t. T_2 as well as no proper T_2 regularly open set w.r.t. T_1.

PROOF Let (X, T_1, T_2) be a pairwise D-space. If A is a proper T_1 regularly open subset of X w. r. t. T_2, then $(\bar{A}^{T_2})^{i}_{1} = A \neq X$ and hence $\bar{A}^{T_2} \neq X$ - a contradiction. Similarly X has no proper T_2 regularly open subset w.r.t. T_1.

We prove a stronger converse that if X has no proper T_1 regularly open set w.r.t. T_2 or no T_2 regularly open set w.r.t. T_1, then (X, T_1, T_2) is a pairwise D-space.

Infact, let (X, T_1, T_2) have no proper T_1 regularly open subset w.r.t. T_2 and let it not be a pairwise D-space. Then there exist nonempty sets $A \in T_1$, $B \in T_2$ such that $A \cap B = \phi$. Then $\bar{A}^{T_2} \cap B = \phi$ and hence $(\bar{A}^{T_2})^{i}_{1} \cap B = \phi$. Then $(\bar{A}^{T_2})^{i}_{1}$ is a proper T_1 regularly open set w.r.t. T_2, which goes against the hypothesis.

Pervin [3] defined a bitopological space (X, T_1, T_2) to be connected if and only if X cannot be expressed as the union of two nonempty disjoint sets A and B such that $(A \cap \bar{B}^{T_2}) \cup (\bar{A}^{T_1} \cap B) = \phi$. He proved that (X, T_1, T_2) is connected if and only if X cannot be expressed as the union of two nonempty disjoint sets A and B such that A is T_1-open and B is T_2-open. Thus we immediately have,

THEOREM 2.9 A pairwise D-space (X, T_1, T_2) is connected.

The next example shows that the converse of the above theorem is false.

EXAMPLE 2.10 Let $X = \{ a, b, c, d \}$, $T_1 = \{ X, \phi, \{ b, c \} \}$ and $T_2 = \{ X. \phi, \{ b \}, \{ d \}, \{ b, d \} \}$.

Then (X, T_1, T_2) is obviously connected but not a pairwise D-space.

THEOREM : 2.11 Pairwise semi-continuous and hence pairwise continuous image of a pairwise D-space is also so.

COROLLARY : 2 12 Every pairwise semi-continuous function (and hence every pairwise continuous function) from a pairwise D-space into any pairwise Hausdorff space is constant.

THEOREM : 2.13 Let $\{ (X_a, T^1_a, T^2_a) : a \in I \}$ be a family of bitopological space (where I is some index set) and (X, T^1, T^2) be their cartesian product, i.e., $X = \pi X_a$ and T^1, T^2 are respectively generated by T^1_a's and T^2_a's. (X, T^1, T^2) is a pairwise D-space if and only if each (X_a, T_a^1, T_a^2) is a pairwise D-space.

PROOF. Since the projection map $P_a : (X, T^i) \rightarrow (X, T_a^i)$ is continuous and open for each $a \in I$ and $i = 1, 2$, by virtue of Theorem 2 11 the necessity follows.

Conversely, suppose (X_a, T^1_a, T^2_a) is a pairwise D-space, for each $a \in I$ and V be a nonempty T^1-open set in X and $x \in V$. Then there exist indices $a_1, a_2, \dots, a_k$ (say) and $U_{a_i} \in T^1_{a_i}$ such that

$$x \in \bigcap_{i=1}^{k} P_{a_i}^{-1} (U_{a_i}) \subset V.$$

Then $X = \bigcap_{i=1}^{k} \left\{ P_{a_i}^{-1} (X_{a_i}) \right\} = \bigcap_{i=1}^{k} \left\{ P_{a_i}^{-1} (\overline{U_{a_i}}^{T^2 a_i}) \right\}$

$$= \overline{P^{-1}_{a_i}(U_{a_i})}T^2 = \overline{P^{-1}_{a_i}(U_{a_i})}T^2 \subset \overline{V}\,T^2 \subset X. \quad \text{Thus } \overline{V}^{T_2} = X.$$

THEOREM 2.14 (a) Let f : $(X, T_1, T_2) \rightarrow (Y, P_1, P_2)$ be one-to-one and pairwise open and (Y, P_1, P_2) is a pairwise D-space, then (X, T_1, T_2) is also so.

(b) Let f : $(X, T_1, T_2) \rightarrow (Y, P_1, P_2)$ be surjective and $\{ f^{-1}(V) : V \in P_i \} = T_i$, for i = 1, 2, then $(X, T_1, T_2,)$ is a pairwise D-space if and only if (Y, P_1, P_2) is so.

REMARK 2.15 The conclusion of Theorem 2.14 (a) may not be true in case either f : $(X, T_1) \rightarrow (Y, P_1)$ or f : $(X, T_2) \rightarrow (Y, P_2)$ is open but not both. This is seen from

EXAMPLE 2.16 Let $X = Y = \{ a, b, c, d \}$,

$T_1 = \{ X, \phi, \{a\}, \{b\}, \{ a, b \}, \{ a, c \}, \{ a, c, b \} \}$, $T_2 = \{ \phi, X, \{b\} \}$,

$P_1 = \{ X, \phi, \{b\}, \{ a, b \} \}$ and $P_2 = \{ X, \phi, \{b\}, \{ b, d \} \}$.

Consider the identity map f : $(X, T_1, T_2) \rightarrow (Y, P_1, P_2)$. Then f : $(X, T_2) \rightarrow (Y, P_2)$ is open but f : $(X, T_1) \rightarrow (Y, P_1)$ is not open. (Y, P_1, P_2) is a pairwise D-space, whereas (X, T_1, T_2) is not so

DEFINITION 2.17 A subset A of (X, T_1, T_2) is called a pairwise D-subset of X if and only if $(A, (T_1)^A, (T_2)_A)$ is a pairwise D-space.

THEOREM 2 18 Let $A \subset Y \subset (X, T_1, T_2)$. Then A is a pairwise D-subset of X if and only if it is a pairwise D-subset of $(Y, (T_1)_Y, (T_2)^Y)$.

PROOF. Easy and left.

THEOREM 2.19 If A is a pairwise D-subset of (X, T_1, T_2), then $\bar{A}^{T_1} \cap \bar{A}^{T_2}$ is also a pairwise D-subset of X

PROOF. If $B = \bar{A}_{T_1} \cap \bar{A}^{T_2}$ is not a pairwise D-subset, then there will exist two nonempty disjoint sets U and V such that $U \in (T_1)_B$ and $V \in (T_2)_B$. Then $U \cap A \ (\neq \phi)$ and $V \cap A \ (\neq \phi)$ are respectively $(T_1)_A$ and $(T_2)_A$ open such that $(U \cap A) \cap (V \cap A) = \phi$ which proves that A is not a pairwise D-space

REMARK 2.20 If A is a pairwise D-subset of (X, T_1, T_2) then neither $\bar{A}_{T_1}$ nor $\bar{A}_{T_2}$ may be pairwise D-subset as is seen from

EXAMPLE 2.21 Let $X = \{a, b, c, d, f\}$,

$T_1, = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and

$T_2 = \{\phi, X, \{a\}, \{f\}, \{a, f\}, \{a, c\}, \{a, c, f\}\}$

and consider the set $A = \{a, c\}$ of the bitopological space $(X, T_1, T_2,)$.

Now, $(T_1)_A = \{A, \phi, \{a\}\}$ and $(T_2)_A = \{A, \phi, \{a\}\}$. A is obviously a pairwise D-subset of X.

Now, $\bar{A}_{T_1} = \{a, c, d, f\} = B$ (say) and $\bar{A}_{T_2} = \{a, b, c, d\} = D$ (say) $(T_1)_B = \{B, \phi, \{a\}, \{a, c\}\}$ and $(T_2)_B = \{B, \phi, \{a\}, \{f\}, \{a, f\}, \{a, c\}, \{a, c, f\}\}$; $(T_1)_D = \{D, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $(T_2)_D = \{D, \phi, \{a\}, \{a, c\}\}$.

Clearly neither B nor D is a pairwise D-subset of X.

REMARK 2.22 Union of two pairwise D-subsets of a. bitopological space may not be a pairwise D-subset as is seen form

EXAMPLE 2.23 Consider (X, T_1, T_2) of Example 2.21. Let $A = \{a, c\}$ and $B = \{b, d\}$ It has been shown in Example 2 21 that A is a pairwise D-subset. Now, $(T_1)_B = \{B, \phi, \{b\}\}$ and $(T_2)_{B'} = \{\phi, B\}$. Clearly B is also a pairwise D-subset of X. But $A \cup B = \{a, b, c, d\}$ is not a pairwise D-subset as shown in Example 2.21.

THEOREM 2.24 If A and B are pairwise D-subsets of (X, T_1, T_2) and if either $(A \cap B^1_2) \neq \phi$ and $(B \cap A^1_2) \neq \phi$, or $A \cap B^1_1 \neq \phi$ and $A^1_1 \cap B \neq \phi$, then $A \cup B$ is a pairwise D-subset of X.

PROOF. Denying the theorem, we get a pair of nonempty sets U and V in X such that $U \in T_1$, $V \in T_2$, $(U \cap V) \cap (A \cup B) = \phi$, but $U \cap (A \cup B) \neq \phi$ and $V \cap (A \cup B) \neq \phi$. Since A and B are pairwise D-subsets, U and V cannot both intersect A or both intersect B. Suppose $U \cap A = \phi$ Since $U \cap (A \cup B) \neq \phi$, we have $U \cap B \neq \phi$ and then $V \cap B = \phi$ and $V \cap A \neq \phi$. If the first assumption of the theorm holds then $U \cap B$ and $A^i_2 \cap B$ are nonempty $(T_1)_B$ and $(T_2)_B$ open subsets of B such that $(U \cap B) \cap (A^i_2 \cap B) \subset U \cap A = \phi$, proving that B is not a pairwise D-subset. If the second assumption of the theorem holds, then $B^i_1 \cap A$ and $V \cap A$ are nonempty and respectively $(T_1)_A$ and $(T_2)_A$ open subsets such that $(B^i_1 \cap A) \cap (V \cap A) \subset V \cap B = \phi$, proving that A is not a pairwise D-subset. The case when $V \cap A = \phi$ can similarly be tackled.

THEOREM 2.25 Union of a chain of pairwise D-subsets of a bitopological space is also a pairwise D-subset.

THEOREM 2.26 Every pairwise D-subset A of a bitopological space is contained in a maximal pairwise D-subsets of the space.

PROOF. Follows from Theorem 2 25 and Zorn's lemma.

COROLLARY 2.27 Every bitopological space is the union of its maximal pairwise D-subsets.

REMARK 2.28 In a bitopological space, maximal pairwise D-subspaces need not be either T_1-open (or T_1-closed) or T_2 open (or T_2-closed). Also maximal pairwise D-subsets need not be disjoint. These are seen from

EXAMPLE 2 29 Consider the bitopological space (X, T_1, T_2) of Example 2.21. It can be checked that the maximal pairwise D-subset containing $A = \{a, c\}$ is $M = \{a. c, d\}$. But M is neither T_1-open nor T_1-closed nor T_2-open nor T_2-closed.

Again the subset $B = \{c, d\}$ is a pairwise D-subset. If B^* denoted the maximal pairwise D-subset of X containing B, then $B^* \subset M$ and hence $B^* \cap M \neq \phi$.

3. PAIRWISE D-CONNECTEDNESS

DEFINITION 3.1 Let (X, T_1, T_2) be a bitopological space. Two nonempty subsets A, B are said to form a pairwise D-seperation of X (where A is T_1-closed and B is T_2-closed) if and only if $X = A \cup B$ and $(A^1_2 \cap B) \cup (A \cap B^1_1) = \phi$. (X, T_1, T_2) is said to be pairwise D-connected if and only if there exists no pairwise D-separation of X.

THEOREM 3.2 A pairwise D-connected space (X, T_1, T_2) is connected but the converse is false.

PROOF. Let X be not connected. Then there exist nonempty T_1-open set A and T_2-open set B such that $X = A \cup B$ and $A \cap B = \phi$. Then C = X-A = B and D = X-B = A are nonempty sets, respectively T_1 closed and T_2-closed and are such that $X = C \cup D$ and $(C^1_2 \cap D) \cup (C \cap D^1{}_1) = (C \cap D) \cup (C \cap D) = B \cap A = \phi$. Hence C and D form a pairwise D-separation of X and the first part of the theorem is proved.

To show that the converse is indeed false, we consider the following example.

Let X = { a, b, c, d }, T_1 = { X, ϕ, { a }, { a, b }, { a, d }, { a, b, d } } and T_2 = { X, ϕ, { a }, { b }, { a, b } }. Then (X, T_1, T_2) is connected.

Now, let A = { b, c } and B = { a, c, d }. Then A, B are respectively T_1-closed and T_2-closed such that $(A^1_2 \cap B) = \{ b \} \cap B = \phi$, $A \cap B^1_1 = \{ b, c \} \cap \{ a, d \} = \phi$ and $X = A \cup B$. Hence A, B form a pairwise D-separation of X and then X is not pairwise D-connected.

THEOREM 3.3 If A, B form a pairwise D-separation of (X, T_1, T_2) and Y is a pairwise D-subset of X, then either $Y \subset A$ or $Y \subset B$.

PROOF. If $Y \not\subset A$ and $Y \not\subset B$, then $Y \cap (X\text{-}A)$ and $Y \cap (X\text{-}B)$ are nonempty disjoint sets in Y, respectively $(T_1)_Y$ open and $(T_2)_Y$ open. Also $[Y \cap (X\text{-}A)] \cap [Y \cap (X\text{-}B)] \subset (X\text{-}A) \cap (X\text{-}B) = X\text{-}A \cup B = X\text{-}X = \phi$. Then Y is not a pairwise D-subset of X. Hence either $Y \subset A$ or $Y \subset B$.

LEMMA 3.4 (X, T_1, T_2) is pairwise D-space if and only if there doesnot exist any pair of nonempty sets A, B respectively T_1-closed and T_2-closed such that $X = A \cup B$,

THEOREM 3.5 (X, T_1, T_2) is a pairwise D-space if and only if X is pairwise D-connected.

PROOF. Necessity follows from Lemma 3.4.

Conversely, let (X, T_1, T_2) be not a pairwise D-space. Then by Theorem 2.8, there exists a nonempty proper T_1 regularly open set A w.r.t. T_2. Then $(\bar{A}^{T_2})'_1 = A$. Let $G = X\text{-}A$ and $H = \bar{A}^{T_2}$ G and H are nonempty sets respectively T_1-closed and T_2-closed. Also, $G'_2 \cap H = (X\text{-}\bar{A}^{T_2}) \cap \bar{A}^{T_2} = \phi$ and $G \cap H'_1 = (X\text{-}A) \cap A = \phi$.

Then G and H form a pairwise D-separation of X, proving that X is not pairwise D-connected.

REMARK 3.6 It is shown in [3] that a space (X, T_1, T_2) is connected iff every pairwise continuous map of X into the bitopological 2-space (Y, D_1, D_2) is constant, where $Y = \{0, 1\}$, $D_1 = \{Y, \phi, \{0\}\}$ and $D_2 = \{Y, \phi, \{1\}\}$. Now by Theorem 3.5 and 2.11 we can easily see that if X is pairwise D-connected, then even every pairwise semi-continuous map of X into the bitopological 2 space is constant. But by taking $X = [a, b, c, d]$, $T_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$ and $T_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ (see converse part of Theorem 3.2) we observe that every pairwise continuous function from (X, T_1, T_2) into the bitopological 2-space is constant, although X is not pairwise D-connected.

NOTE This paper was communicated to Kyungpook Mathematical Journal in early January, 1983 Since the paper has neither been accepted nor rejected till now in spite of repeated reminders, the author has preferred to publish, in the departmental journal, a shortened version of the paper deleting most of the proofs.

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REFERENCES

[1] Kelly, J. C. ; Bitopological spaces, Proc, London Math. Soc., 13 (1963). 71-89.

[2] Norman Levine, Dense topologies, Amer, Math. Monthly, 75 (1968), 847-852.

[3] W. J. Pervin ; Connectedness in bitopological spaces, Indag, Math , 29 (1967), 369-372.

[4] I. L. Reilly ; On bitopological separation properties, Nanta Math., 5 No. 2 (1972), 14-25.

[5] R. Shantha ; Problems relating to some basic concepts in bitopological spaces, Ph D. Thesis submitted to the Calcutta University.

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A SYSTEM OF AFFINE CONNECTIONS ON A RIEMANNIAN MANIFOLD

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O. In 1950, Sen [4] constructed an algebraic system generated by a single element. In course of his investigation he obtained a cyclic sequence under certain conditions. Later Chaki [1] started with the coefficients of an arbitrary affine connection in a Riemannian space and constructed a cyclic sequence of twelve distinct affine connections. In 1961 Gupta [2] obtained several interesting results regarding the sequence obtained by Chaki.

In section one of the present paper, the sequence of affine connections constructed by Chaki has been generalized on a Riemannian manifold and the corresponding sequence of torsion tensors has been constructed. A few interesting results regarding this sequence have been obtained. Section two deals with the covariant derivatives of the Riemann metric with respect to the affine connections of the sequence. Finally, in section three, the curvature tensors of the connections of the sequence have been calculated.

1. Let M be a Riemannian manifold with Riemann metric g and let ∇ be an arbitrary affine connection defined on M. Write

$$a = g(\nabla_Y Z, X) \tag{1.1}$$

where X, Y, Z, ... are differentiable vectorfields on M. The associate a^* and the conjugate a' of a are defined by

$$\left| \begin{array}{l} a^* = g(\nabla_Y Z, X) + (\nabla_Y g)(Z, X) \\ \\ a' = g(\nabla_Z Y, X) \end{array} \right. \tag{1.2}$$

Evidently, $a^{**} = a'' = a$. The connection a is said to be self-associate or self-conjugate according as $a^* = a$ or $a' = a$ respectively.

Let $H(Y, Z)$ be a differentiable tensorfield of type (1.2) on M. Define an affine connection $\overset{1}{\nabla}$ on M by the relation

$$(1.3) \quad \overset{1}{\nabla}_Y Z = \nabla_Y Z + H(Y, Z)$$

and write

$$(1.4) \quad d_1 = g(\overset{1}{\nabla}_Y Z, X)$$

Starting with d_1 a sequence of affine connections can be obtained by forming successively the associate and conjugate of d_1, and the elements of this sequence may be denoted by d_1, d_2, d_3, ... etc.
Thus

$$d_2 = d_1^* = g(\overset{2}{\nabla}_Y Z, X) = g(\overset{1}{\nabla}_Y Z, X) + (\overset{1}{\nabla}_Y g)(Z, X)$$

$$d_3 = d_1^{*\prime} = d'_2 = g(\overset{3}{\nabla}_Y Z, X) = g(\overset{2}{\nabla}_Z Y, X)$$

and so on. Proceeding in this way a cyclic sequence of twelve affine connections can be obtained which is as follows :

$$(\mathrm{I}) \quad \left| \begin{aligned}
d_1 &= a + \gamma \\
d_2 &= a + \alpha - \delta \\
d_3 &= a' + \alpha' - \delta' \\
d_4 &= a + \alpha - \lambda + \beta + \varepsilon' \\
d_5 &= a' + \alpha' - \lambda + \beta' + \varepsilon \\
d_6 &= a + \alpha + \alpha' - \lambda + \beta + \beta' - \gamma' \\
d_7 &= a' + \alpha + \alpha' - \lambda + \beta + \beta' - \gamma \\
d_8 &= a' + \alpha' - \lambda + \beta + \beta' + \delta \\
d_9 &= a + \alpha - \lambda + \beta + \beta' + \delta' \\
d_{10} &= a' + \alpha' + \beta' - \varepsilon' \\
d_{11} &= a + \alpha + \beta - \varepsilon \\
d_{12} &= a' + \gamma'
\end{aligned} \right.$$

where

$$(\mathrm{II})\quad \left|\begin{array}{ll} \alpha = (\nabla_Y g)(Z, X), & \alpha' = (\nabla_Z g)(Y, X) \\ \lambda = (\nabla_X g)(Y, Z) = \lambda' & \\ \beta = g(\nabla_Y X - \nabla_X Y, Z), & \beta' = g(\nabla_Z X - \nabla_X Z, Y) \\ \gamma = g(H(Y, Z), X), & \gamma' = g(H(Z, Y), X) \\ \delta = g(H(Y, X), Z), & \delta' = g(H(Z, X), Y) \\ \epsilon = g(H(X, Y), Z\ , & \epsilon' = g(H(X, Z), Y) \end{array}\right.$$

It is interesting to see that the affine connection 'd' defined by

(1.5) $2d = d_k + d_{k+6}$, $k = 1, \dots, 6$

is both self-associate and self-conjugate, that is,

$$d^* = d \text{ and } d' = d$$

and it is given by

(1.6) $2d = a + a' + \alpha + \alpha' - \lambda + \beta + \beta'$

If $\overset{k}{T}(Y, Z) = \overset{k}{\nabla}_Y Z - \overset{k}{\nabla}_Z Y - [Y, Z]$, $k = 1, 2, \dots, 12$,

denotes the torsion tensor of the affine connection $\overset{k}{\nabla}$, where $d_k = g(\overset{k}{\nabla}_Y Z, X)$, then a straightforward calculation shows that

$$(1.7)\quad \left|\begin{array}{l} \overset{2}{T}(Y,Z) = \overset{9}{T}(Y,Z) = -\overset{3}{T}(Y,Z) - 2[Y,Z] = -\overset{8}{T}(Y,Z) - 2[Y,Z] \\ \overset{4}{T}(Y,Z) = \overset{11}{T}(Y,Z) = -\overset{5}{T}(Y,Z) - 2[Y,Z] = -\overset{10}{T}(Y,Z) - 2[Y,Z] \\ \overset{6}{T}(Y,Z) = \overset{1}{T}(Y,Z) = -\overset{7}{T}(Y,Z) - 2[Y,Z] = -\overset{12}{T}(Y,Z) - 2[Y,Z] \end{array}\right.$$

Theorem (1.1) : The torsion tensors of the sequence (I) satisfy

$$(1.8)\qquad \overset{k}{T}(Y,Z) + \overset{k+9}{T}(Y,Z) = -2[Y,Z]$$

The proof is immediate.

It is easy to see that the torsion tensor $\tau\,(Y, Z)$ of the connection 'd' defined in (1.5) is

$$(1.9)\quad \tau(Y,Z) = \overset{k}{T}(Y,Z) + \overset{k+6}{T}(Y,Z) = -2\,[Y,Z]$$

It is known that every Riemannian manifold admits a Riemann connection or Levi-Civita connection which is a metric connection and whose torsion tensor is zero. If $\bar{\nabla}$ denotes this L. C. connection, then [3].

$$2g(\bar{\nabla}_Y Z, X) = Yg(Z,X) + Zg(Y,X) - Xg(Y,Z)$$
$$+\, g([X,Y],Z) + g([X,Z],Y) + g([Y,Z],X)$$

Writing $d = g(D_Y Z, X)$ it can be seen, by virtue of (1.9),

$$(1.10)\quad 2g(\bar{\nabla}_Y Z, X) = 2g(D_Y Z, X) - g(\tau(X,Y),Z) - g(\tau(X,Z),Y)$$
$$-\, g(\tau(Y,Z),X)$$

Theorem (1.2) : In terms of the connections of the sequence (I), the L.C. connection of the Riemann manifold is given by (1.10).

2. In this section symmetric covariant derivatives of the Riemann metric g have been dealt with.

Covariant derivative of the Riemann metric g is said to be symmetric with respect to an affine connection ∇ if

$$(2.1)\quad (\nabla_Y g)(Z,X) = (\nabla_Z g)(Y,X)$$

for all differentiable vectorfields on M. In this section the above relation will be considered to be true.

If $(\overset{k}{\nabla}_Y g)(Z,X) = Yg(Z,X) - g(\overset{k}{\nabla}_Y Z, X) - g(Z, \overset{k}{\nabla}_Y X)$, $k = 1, 2, \ldots, 12$,

denote the covariant derivative of the Riemann metric g with respect to the affine connection $\overset{k}{\nabla}$, then

$$(2.2)\quad \left|\begin{array}{l} (\overset{1}{\nabla}_Y g)(Z, X) = -(\overset{2}{\nabla}_Y g)(Z, X\,, = -(\overset{7}{\nabla}_Y g)(Z, X) = \\ (\overset{8}{\nabla}_Y g)(Z, X) \\[1em] (\overset{3}{\nabla}_Y g)(Z, X) = -(\overset{4}{\nabla}_Y g)(Z, X) = -(\overset{9}{\nabla}_Y g)(Z, X) = \\ (\overset{10}{\nabla}_Y g)(Z, X) \\[1em] (\overset{5}{\nabla}_Y g)(Z, X) = -(\overset{6}{\nabla}_Y g)(Z, X) = -(\overset{11}{\nabla}_Y g)(Z, X) = \\ (\overset{12}{\nabla}_Y g)(Z, X) \end{array}\right.$$

From (2.2), (2.1) and (II) it can be seen that

(i) $\overset{1}{\nabla} g, \overset{2}{\nabla} g, \overset{7}{\nabla} g, \overset{8}{\nabla} g$ are symmetric if and only if
$\delta + \gamma - \varepsilon - \varepsilon' = 0$;

(ii) $\overset{3}{\nabla} g, \overset{4}{\nabla} g, \overset{9}{\nabla} g, \overset{10}{\nabla} g$ are symmetric if and only if
$2\beta + \beta' + \tau - \gamma + \varepsilon' - \delta' + \gamma' = 0$;

(iii) $\overset{5}{\nabla} g. \overset{6}{\nabla} g, \overset{11}{\nabla} g, \overset{12}{\nabla} g$ are symmetric if and only if
$2\beta + \beta' + \tau - \varepsilon + \delta + \delta' - \gamma' = 0$

Theorem (2.1) : In view of the symmetry of the covariant derivatives of the Riemann metric g, the elements of the sequence (I) can be divided into three classes $\{ d_1, d_2, d_7, d_8 \}$, $\{ d_3, d_4, d_9, d_{10} \}$ and $\{ d_5, d_6, d_{11}, d_{12} \}$.

Theorem (2.2) : If the covariant derivative of g is symmetric with respect to the connections in any two of the above classes, then it is symmetric with respect to the connections of the remaining class also.

3. The curvature tensor $K(X, Y)Z$ of the affine connection ∇ is given by

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \tag{3.1}$$

and its fully covariant form $'K(X, Y, Z, W)$ is given by

$$'K(X, Y, Z, W) = g(K(X, Y)Z, W) \tag{3.2}$$

If $'\overset{\iota}{K}(X, Y, Z, W)$, $\iota = 1, 2, \dots, 12$ denote the fully covariant curvature tensors of the affine connections of the sequence (I , then they form the following sequence :

$$\text{(III)}\quad \left|\begin{aligned}
'\overset{1}{K}(X, Y, Z, W) &= {}'K(X, Y, Z, W) + g((\nabla_X H)(Y, Z), W) \\
&\quad - g((\nabla_Y H)(X, Z), W) \\
&\quad + g(H(T(X, Y), Z), W) + g(H(X, H(Y, Z)), W) \\
&\quad - g(H(Y, H(X, Z)), W) \\
'\overset{2}{K}(X, Y, Z, W) &= -{}'\overset{1}{K}(X, Y, W, Z) \\
2\,'\overset{3}{K}(X, Y, Z, W) &= 2\,'\overset{2}{K}(X, Y, Z. W) + g((\overset{3}{\nabla}_Y \overset{3}{T})(X, Z), W) \\
&\quad - g((\overset{3}{\nabla}_X \overset{3}{T})(Y, Z), W) - \tfrac{1}{2} g(\overset{3}{T}(\overset{3}{T}(X, Y), Z), W) \\
'\overset{4}{K}(X, Y, Z, W) &= -{}'\overset{3}{K}(X, Y, W, Z) \\
2\,'\overset{5}{K}(X, Y, Z, W) &= 2\,'\overset{4}{K}(X, Y, Z, W) + g((\overset{5}{\nabla}_Y \overset{5}{T})(X, Z), W) \\
&\quad - g((\overset{5}{\nabla}_X \overset{5}{T})(Y, Z), W) - \tfrac{1}{2} g(\overset{5}{T}(\overset{5}{T}(X, Y), Z), W) \\
'\overset{6}{K}(X, Y, Z, W) &= -\,,\overset{5}{K}(X, Y, W, Z) \\
2\,'\overset{7}{K}(X, Y, Z, W) &= 2\,'\overset{6}{K}(X, Y, Z, W) + g((\overset{7}{\nabla}_Y \overset{7}{T})(X, Z), W) \\
&\quad - g((\overset{7}{\nabla}_X \overset{7}{T})(Y, Z), W) - \tfrac{1}{2} g(\overset{7}{T}(\overset{7}{T}(X, Y), Z), W) \\
'\overset{8}{K}(X, Y, Z, W) &= -{}'\overset{7}{K}(X, Y, W, Z)
\end{aligned}\right.$$

$$\left|\begin{aligned}
&2\,'\overset{9}{K}(X,Y,Z,W) = 2\,'\overset{8}{K}(X,Y,Z,W) + g((\overset{9}{\nabla}_Y \overset{9}{T})(X,Z),W) \\
&\qquad - g(\overset{9}{\nabla}_X \overset{9}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{9}{T}(\overset{9}{T}(X,Y),Z),W) \\
&'\overset{10}{K}(X,Y,Z,W) = -\,'\overset{11}{K}(X,Y,W,Z) \\
&2\,'\overset{11}{K}(X,Y,Z,W) = 2\,'\overset{10}{K}(X,Y,Z,W) + g((\overset{11}{\nabla}_Y \overset{11}{T})(X,Z),W) \\
&\qquad - g((\overset{11}{\nabla}_X \overset{11}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{11}{T}(\overset{11}{T}(X,Y),Z),W) \\
&'\overset{12}{K}(X,Y,Z,W) = -\,'\overset{11}{K}(X,Y,W,Z)
\end{aligned}\right.$$

Assuming that

$$(3.3)\quad \left|\begin{aligned}
&\overset{1}{T}(Y,Z) = \overset{2}{T}(Y,Z) = \overset{4}{T}(Y,Z) = \overset{6}{T}(Y,Z) = \overset{9}{T}(Y,Z) = \overset{11}{T}(Y,Z) = 0 \\
&\text{and consequently,} \\
&\overset{3}{T}(Y,Z) = \overset{5}{T}(Y,Z) = \overset{7}{T}(Y,Z) = \overset{8}{T}(Y,Z) = \overset{10}{T}(Y,Z) = \overset{12}{T}(Y,Z) = \\
&-2[Y,Z]
\end{aligned}\right.$$

from sequence (III) we find

(i) $'\overset{l}{K}(X,Y,W,Z) + {'\overset{l+1}{K}}(X,Y,Z,W) = 0$ for $l = 1, 3, 5, 7, 9, 11,$

(ii) $$2\{'\overset{l+1}{K}(X,Y,Z,W) - {'\overset{l}{K}}(X,Y,Z,W)\} = g((\overset{l+1}{\nabla}_Y \overset{l+1}{T})(X,Z),W) - g((\overset{l+1}{\nabla}_X \overset{l+1}{T})(Y,Z),W) - \tfrac{1}{2}\, g(\overset{l+1}{T}(\overset{l+1}{T}(X,Y),Z),W)$$

for $l = 2, 4, 6,$

(iii) $2\{'\overset{l}{K}(X,Y,Z,W) - {'\overset{l+1}{K}}(X,Y,Z,W) = g((\overset{l}{\nabla}_Y \overset{l}{T})(X,Z),W)$

$$- g((\overset{l}{\nabla}_X \overset{l}{T})(Y, Z), W) \quad - \tfrac{1}{2} g(\overset{l}{T}(\overset{l}{T}(X, Y), Z \;, W$$

for $l = 8, 10. 12.$

Writting

$t(Y, Z) = 2[Y, Z]$, the common value of $\overset{3}{T}, \overset{5}{T}, \overset{7}{T}, \overset{8}{T}, \overset{10}{T}, \overset{12}{T}$ and

$$\overset{1}{D} = \overset{3}{\nabla} + \overset{7}{\nabla} - \overset{10}{\nabla}, \overset{2}{D} = \overset{5}{\nabla} - \overset{8}{\nabla} - \overset{12}{\nabla}$$

we find from sequence (III), that

$$(3.4) \quad g((\overset{1}{D}_Y t)(X, Z), W) - g((\overset{1}{D}_X t)(Y, Z), W) - \tfrac{3}{2} g(t(t(X, Y), W)$$

$$= g((\overset{2}{D}_Y t)(X, W), Z) - g((\overset{2}{D}_X t)(Y, W), Z) - \tfrac{3}{2} g(t(t(X, Y), W), Z)$$

REFERENCES

[1] Chaki, M. C. : Some formulas in a Riemannian space. "Ann. della Scuola normal Superiore di Pisa " Serie III, 10, (1956).

[2] Gupta, B. : Some theorems in a Riemannian space with an added affine connection. "Ren de la Fac. des sci. de Uni. d' Istambul," Serie A, 26, (1961).

[3] Kobayshi. S. and Nomizu, K. : Foundations of differential geometry. Vol. I. Interscience Publishere (1969).

[4] Sen, R. N. : On an algebraic system generated by a single element and its application in Riemannian geometry. "Bul. Cal. Math. Soc." 42 (1950).

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SOME RESULTS ON FIXED POINT THEOREM IN COMPACT METRIC SPACE.

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§ 1 : INTRODUCTION :—

Let (X, d) metric space and T : X → X be an operator. T is said to be a contraction mapping if it satisfies

1) $d(Tx, Ty) \leqslant qd(x, y), \forall\, x, y \in X, 0 < q < 1.$

and T is said to be a contractive mapping if it satisfies

2) $d(Tx, Ty) < qd(x, y), \forall\, x, y \in X, q=1.$

A contraction mapping is contractive but the converse is clearly not true.

If (X, d) be a complete metric space and T : X → X satisfies (1) then by Banach's fixed point theorem, T has a unique fixed point but if T : X → X satisfy (2) then T need not have a fixed point.

Again if (X, d) is a compact matric space and T : X → X satisfies (2) then T has a unique fixed point. Moreover, it follows from this fact that on a compact metric space the notions contraction and of contractive mappings coincide.

On compact metric space, many authors proved fixed point theorems using various contractive type of mappings. Of them, B. Fisher [3] proved the following theorem using contractive type mapping yielding a unique fixed point.

THEOREM – A : (Fisher)

If T is a continuous mapping of a compact metric sapce (X, d) into itself such that
$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)]$

for all x, $y \in X$ with $x \neq y$, then T has a unique fixed point.

S. K. Chatterjea [1] proved the following theorem involving G. Hardy and T. Rogers [4] type contractive mapping yielding a unique fixed point.

THEOREM ; B : (Chatterjea) -

If T is a continuous mapping of a compact metric space (X, d) into itself such that $d(Tx, Ty) < a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y\ Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$ for all distinct $x, y \in X$, $a_i \geqslant \Phi$ ($i = 1, 2, \dots, 5$) and $\sum_1^5 a_i = 1$, then T has a unique fixed point.

In fact, the theorem of Fisher is a particular case of the theorem due to Chatterjea.

In this paper we have tried to make extension of the results of Fisher and Chatterjea.

δ 2 : FIXED POINT THEOREMS :—

In this section we have proved the following theorems.

THEOREM-1 :

Let (X, d) be a compact metric space and T_1, T_2 be a pair of continuous self mappings defined on X for which there exists non-negative real numbers q_j ($j=1, 2, \dots, 5$) with $\sum_{j=1}^{5} q_j = 1$ such that $d(T_1 x, T_2 y) < q_1 d(x, y) + q_2 d(x, Tx) + q_3 d(y, Ty) + q_4 d(x, Ty) + q_5 d(y, Tx)$ for all distinct $x, y \in X$, then T_1 and T_2 have a unique common fixed point in X.

PROOF : We first define a function $F : X \to R_+$ as follows :

$F(X) = d(x, T_1 x)$, $\forall\ x \in X$. Clearly F is continuous.

Let $F(z) = \inf\{F(x) : x \in X\}$, which must exist since X is compact.

Now, if possible, suppose that $z \neq T_1 z$ and $T_2 T_1 z \neq T_1 z$. Then $F(T_2 T_1 z) = d(T_2 T_1 z, T_1 T_2 T_1 z) = d(T_1 T_2 T_1 z, T_2 T_1 z)$

$< q_1 d(T_2 T_1 z, T_1 z) + q_2 d(T_2 T_1 z, T_1 T_2 T_1 z) + q_3 d(T_1 z, T_2 T_1 z)$
$+ q_4 d(T_2 T_1 z, T_2 T_1 z) + q_5 d(T_1 z, T_1 T_2 T_1 z)$

$\Rightarrow (1 - q_2 - q_5) F(T_2 T_1 z) < (1 - q_2 - q_4) d(T_1 z, T_2 T_1 z)$ ......(i)

Again, by symmetry, we have

$(1 - q_3 - q_4) F(T_2 T_1 z) < (1 - q_3 - q_5) d(T_1 z, T_2 T_1 z)$ ..... (ii)

Adding (i) and (ii) we get

$\{2-(q_2 + q_3 + q_4 + q_5)\} F(T_2 T_1 z) < \{2-(q_2 + q_3 + q_4 + q_5)\} d(T_1 z, T_2 T_1 z)$

$\Rightarrow F(T_2 T_1 z) < d(T_1 z, T_2 T_1 z)$ .... (I)

Now, $d(T_1 z, T_2 T_1 z) < q_1 d(z, T_1 z) + q_2 d(z, T_1 z) + q_3 d(T_1 z, T_2 T_1 z)$
$+ q_4 d(z, T_2 T_1 z) + q_5 d(T_1 z, T_1 z)$

$\Rightarrow (1-q_3-q_4) d(T_1 z, T_2 T_1 z) < (1-q_3-q_5) F(z)$.

By symmetry, we have

$(1-q_2-q_5) d(T_1 z, T_2 T_1 z < (1-q_2-q_4) F(z)$.

Adding these two we get

$\{2-(q_2 + q_3 + q_4 + q_5)\} d(T_1 z, T_2 T_1 z) < \{2-(q_2 + q_3 + q_4 + q_5)\} F(z)$

$\Rightarrow d(T_1 z, T_2 T_1 z) < F(z)$ .... (II)

From (I) and (II) we get $F(T_2 T_1 z) < F(z)$ — which is a contradiction.

Hence $T_2 T_1 z = T_1 z$ and $T_1 z = z$.

Therefore, $T_2 z = z$.

Consequently, z is a common fixed point of T_1 and T_2 both.

We shall now show the unicity of the fixed point z.

Suppose that $u (\neq z) \in X$ is another common fixed point of T_1 and T_2.

Then, $d(z, u) = d(T_1 z, T_2 u)$

$< q_1 d(z, u) + q_2 d(z, T_1 z) + q_3 d(u, T_1 u) + q_4 d(z, T_2 u) + q_5 d(u, T_1 z)$

$= (q_1 + q_4 + q_5) d(z, u) = (1-q_2-q_3) d(z, u)$

$\Rightarrow (q_2 + q_3) d(z, u) < 0$—a contradiction.

So, $d(z, u) = O \Rightarrow z = u$.

Consequently, z is the unique common fixed point of T_1 and T_2.

This completes the proof of the theorem.

NOTE :

I) If we take $T_1 = T_2 = T$ and $q_1 = q_4 = q_5 = O$ and $q_2 = q_3 = \frac{1}{2}$ then we get theorem-A.

II) If we take $T_1 = T_2 = T$ we get theorem-B.

THEOREM - 2 : If T be a mapping of a compact metric space (X, d) into itself such that for some fixed positive integer m, T^m satisfies the inequality.

$$d(T^m x, T^m y) < q_1 d(x, y) + q_2 d(x, T^m x) + q_3 d(y, T^m y) + q_4 d(x, T^m y) + q_5 d(y, T^m x)$$

for all $x, y \in X$, $x \neq y$ where $q_i \geqslant O$ with $\sum_{i=1}^{5} q_i = 1$ and if T^m is continuous then T has a unique fixed point in X.

PROOF : In view of Theorem-B, T^m has a unique fixed point z in X. Now, since $T^m z = z$, we have

$Tz = T(T^m z) = T^m(Tz) = Tz \Rightarrow Tz$ is a fixed point of T^m in X, but T^m has a unique fixed point z in X.

Hence, $Tz = z$.

Therefore, z, being unique, is a unique fixed point of T in X.

NOTE :

We can show, by an example, that Theorem-2 is stronger than Theorem-B.

EXAMPLE :- We take $X = R$ and we define $T : R \to R$ as follows ;-

$$\begin{aligned} Tx &= 0 \text{ if } 0 \leqslant x < \tfrac{1}{2} \\ &= \tfrac{1}{2} \text{ if } \tfrac{1}{2} \leqslant x \leqslant 1. \end{aligned}$$

So, $T^2 x = 0, \forall x \in [0, 1]$.

Clearly, 0 is a unique fixed point of T^2 and also of T.

It is easy to verify that the inequality of Theorem-2 is satisfied by T^2 for $m = 2$ but not by T.

Following is the more general theorem :

THEOREM-3 : If T_1 and T_2 be two mappings on a compact metric space (X, d) into itself such that for some positive integers r, s, T_1^r and T_2^s satisfy the inequality

$$d(T_1^r x, T_2^s y) < q_1 d(x, y) + q_2 d(x, T_1^r x) + q_3 d(y, T_2^s y) +$$

$$q_4 d(x, T_2^s y) + q_5 d(y, T_1^r x)$$

for all $x, y \in X$ with $x \neq y$, where $q_i \ (i = 1, 2, \ldots 5) \geqslant 0$, $\sum_{i=1}^{5} q_i = 1$,

then T_1 and T_2 have a unique common fixed point in X.

PROOF : Let $x \in X$ be an arbitrary element. We now define a sequence $\{x_n\} \subseteq X$ as follows :

$$x_1 = T_1^r x, \ x_2 = T_2^s x_1, \ x_3 = T_1^r x_2, \ldots\ldots\ldots, x_{2n} = T_2^s x_{2n-1},$$

$$x_{2n+1} = T_1^r x_{2n}, \ldots\ldots\ldots$$

Since, X is compact and $x_n \in X$, $\forall n \geqslant 1$, we can always select a subsequence $\{x_{n_i}\}$ from the sequence $\{x_n\}$ such that $\lim_{i \to \infty} x_{n_i} = u$ for some $u \in X$.

If possible, suppose that $T_1^r u \neq u$.

Then $d(u, T_1^r u) \leqslant d(u, x_{2n_i}) + d(x_{2n_i}, T_1^r u)$

$$= d(u, x_{2n_i}) + d(T_1^r u, T_2^s x_{2n_i-1})$$

$$< d(u, x_{2n_i}) + q_1 d(u, x_{2n_i-1}) + q_2 d(u, T_1^r u)$$

$$+ q_3 d(x_{2n_i-1}, T_2^s x_{2n_i-1}) + q_4 d(u, T_2^s x_{2n_i-1}) + q_5 d(x_{2n_i-1}, T_1^r u)$$

$=d(u, x_{2n_i}) + q_1 d(u, x_{2n_i-1}) + q_2 d(u, T_1^r u) + q_3 d(x_{2n_i-1}\ x_{2n_i}) +$

$q_4 d\ d(u, x_{2n_i}) + q_5 d(x_{2n_i-1}, T_1^r u)$.

Now proceeding to the limit as $i \to \infty$, we have

$(1 - q_2 - q_5)\ d(u, T_1^r u) < 0$ —— which is a contradiction.

Hence, we have $T_1^r u = u$

Likewise, we can show that $T_2^s u = u$

We now show the unicity of the fixed point u.

Suppose, now, that $p(\neq u) \in X$ is another common fixed point of T_1^r and T_2^s both. Then

$$d(p, u) = d(T_1^r p, T_2^s u) < q_1 d(p, u) + q_2 d(p, T_1^r p) + q_3 d(p, T_2^s p) + q_4 d(u, T_2^s p) + q_5 d(p, T_1^r u)$$

$$= (q_1 + q_4 + q_5) d(p, u) = (1 - q_2 - q_3) d(p, u)$$

$\Rightarrow (q_2 + q_3)\ d(p, u) < 0$ —— which is a contradiction, and thereby yielding $p = u$.

Consequently, u is a common fixed point of T_1^r and T_2^s in X.

Now, since, $T_1^r u = u$, we have $T_1 u = T_1(T_1^r u) = T_1^r (T_1 u)$ –
this implies that $T_1 u$ is another fixed point of T_1^r in X, but T_1^r has only one fixed point u in X.

Hence, this enables to conclude that $T_1 u = u$.

Similarly, we can show that $T_2 u = u$.

If z is another common fixed point of T_1 and T_2 in X, then clearly it is a common fixed point of T_1^r and T_2^s in X also. This yields $u = z$.

Hence, u is the unique common fixed point of T_1 and T_2 in X.
This completes the proof of the theorem.

§ 3 :- Let (X,d) be a metric space. Let $T : X \to X$ be a mapping which is said to possess the property (A) if it satisfies

$d(Tx,Ty) \leqslant ad(x,Tx) + bd(y,Ty) + cd(x,y)$

for all $x,y \in X$, $a,b,c \geqslant 0$, $a+b+c \leqslant 1$.

According to Kannan (1972) [5], an operator $T : X \to X$ is said to have the property (B) on $E \subset X$ if for every closed subset F of E containing more than one element and is self mapped by T, there exists an $x \in F$ such that

$$d(x, Tx) < \sup_{y \in F} d(y, Ty).$$

Kannan (1972) [5] studied this property (B) of an operator T in connection with the existence of fixed point. With the above notions, M. K. Chakraborty [2] proved the following

THEOREM-C :-

Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping having properties (A) and (B) over X. Let for any non-empty subset E of X which is mapped into itself by T, $p_n \to p$ imply $Tp_n \to p$ for any sequence $\{p_n\} \subset E$. Then T has a fixed point in X provided that $b \neq 1$. The fixed point is unique if $c \neq 1$.

Let (X, d) be a metric space. An operator $T : X \to X$ is said to satisfy Hardy and Regers [4] type of contraction if $d(Tx, Ty) \leqslant q_1 d(x, y) + q_2 d(x, Tx) + q_3 d(y, Ty) + q_4 d(x, Ty) + q_5 d(y, Tx)$ holds for all $x, y \in X$ where q_i ($i = 1, 2, 3, \ldots, 5$) $\geqslant 0$, $\sum_{i=1}^{5} q_i < 1$. ——(i)

We shall prove the following theorem in fixed point in compact metric space by weakening the condition (i) above, by allowing $\sum_{i=1}^{5} q_i = 1$ --- (ii) which extends the theorem-C.

We shall call (i) and (ii) together the property ($*$).

THEOREM :-1 :Let (X, d) be compact metric space and let T be a self mapping defined on X satisfying the properties ($*$) and (B) over X. Then if T be such that for any non-empty subset E of X, mapped into itself by T, $p_n \to p$ imply $Tp_n \to p$ where $\{ p_n \} \subset E$, then T has a fixed point in X. This fixed point will be unique provided that $q_2 > o$ or $q_3 > o$.

PROOF : We consider the space X(K) of all non-empty closed subsets K of X for which $T : K \to K$.

We now define a partial ordering in X(K) by the following rule :

$$K_{\alpha_1} \prec K_{\alpha_2}, \; K_{\alpha_1} \underset{\neq}{\supset} K_{\alpha_2}.$$

With this definition, the space X(K) will be a partially ordered set and hence by Kuratowski-Zorn lemma, there must exists a minimal element in X(K) which will be non-empty closed and invariant under T. Let it be denoted by K.

We now propose to show that K contains only one element.

If possible, suppose, K consists of more than one element. Then by the property (B), there exists an $x_o \in K$ such that $d (x_o, Tx_o) = r < \sup_{y \in K} d (y, Ty)$ .... (1)

Now consider the set $K_1 = \{ x \in K : d (x, Tx) \leqslant r \}$. Then by (1), K_1 is a non-empty proper subset of K. We shall first show that $T : K_1 \to K_1$. Take any $x \in K_1$.

Now, $d (Tx, T^2 x) = d (Tx, TTx) \leqslant q_1 d (x, Tx) + q_2 d (x, Tx) + q_3 d (Tx, TTx) + q_4 d (x, TTx) + q_5 d (Tx, Tx)$

$\Rightarrow (1-q_3-q_4) d (Tx, T^2x) \leqslant (1-q_3-q_5) d (x, Tx).$ ....(2)

By symmetry, we can write

$(1-q_2-q_5) d (Tx, T^2x) \leqslant (1-q_2-q_4) d (x, Tx).$ ....(3)

Adding (2) and (3) and cancelling the non-negative constant terms form both sides, we get

$d (Tx, T^2x) \leqslant d (x, Tx) \leqslant r \Rightarrow Tx \in K_1.$

This shows that K_1 is mapped into itself by T.

We shall now show that K_1 is closed.

Choose any sequence $\{ s_n \} \subset K_1$ such that $s_n \rightarrow s$ as $n \rightarrow \infty$ where $s \in K$.

Hence by assumsion, $Ts_n \rightarrow s$.

$$\begin{aligned} \text{Now, } d (s, Ts) &\leqslant d (s, Ts_n) + d (Ts_n, Ts) \\ &\leqslant d (s, Ts_n) + q_1 d (s_n, s) q_2 + d (s_n, Ts_n) + q_3 d (s, Ts) \\ &\qquad + q_4 d (s_n, Ts) + q_5 d (s, Ts_n) \\ &\leqslant (1 + q_5) d (s, Ts_n) + q_1 d (s_n, s) + q_2 r + q_3 d (s, Ts) + \\ &\quad q_4 d (s_n, Ts \;. \end{aligned}$$

Now passing to the limit as $n \rightarrow \infty$ we have after rearrangement

$(1-q_3-q_4) d (s, Ts) \leqslant q_2 r,$ ....(4)

In a similar way we can write by interchanging the roles of s_n and s,

$(1-q_2-q_5) d (s,Ts) \leqslant q_3 r.$ ....(5)

Adding these two we get

$\{2-(q_2+q_3+q_4+q_5)\} d (s,Ts) \leqslant (q_2+q_3) r$

$$\Rightarrow d (s,Ts) \leqslant \frac{(q_2+q_3)}{\{2-(q_2+q_3+q_4+q_5)\}} r \leqslant r, \Rightarrow \quad s \in K_1.$$

Hence K_1 is closed.

Thus K_1 is a non-empty closed proper subset of K with $T : K_1 \rightarrow K_1$ and so $K_1 \in X(K)$. This contradicts the minimality of K in X (K). Hence K contains only one point, which is a fixed point of T in X.

Now suppose that $u \in K$ is this fixed point of T.

Let $v\ (\neq u) \in K$ be another fixed point of T. Then

$$d(u,v) = d(Tu,Tv) \leqslant q_1 d(u,v) + q_2 d(u,Tu) + q_3 d(v,Tv) + q_4 d(u,Tv) + q_5 d(v,Tu)$$

$$\leqslant (1-q_2-q_3)\, d(u,v),$$

$$\Rightarrow (q_2+q_3)\, d(u,v) \leqslant 0 \Rightarrow d(u,v) = 0 \text{ as } q_2 > 0 \text{ or } q_3 > 0,$$

$$\Rightarrow u = v.$$

Consequently, T has a unique fixed point in X,

This completes the proof of the theorem.

NOTE : It is to be noted here that if we put $q_4=0, q_5 = 0$, $q_2=a, q_3=b$ and $q_3=c$ then the theorem-C follows at once.

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REFERENCES :

1) Chatterjea, S. K. : "Some Contraction Mappings",
Anal. Stintifice. ale Universite 'Al I Cuze' din Iasi
Tom XXV, S. Ia 1979, f.2, pp. 283-286.

2) Chakraborty, M. K. : "Some Results on Fixed Point of Operators"
Indian J. Pure Appl. Math., Vol. 6, No. 7 (1975), pp. 809-813.

3) Fisher, B. : "A Fixed Point Mapping",
Bull. Cal. Math. Soc., 68(1975), pp. 265-266.

4) Hardy, G. E. & Rogers, T. D. : "A Generalisation of a Fixed Point Theorem of Reich", Canad. Math. Bull, Vol. 16(2), 1973, pp. 201-206.

5) Kannan, R. : "Some Results on Fixed Points-IV", Fundamenta Mathematicae LXXIV (1972), pp.181-187.

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ON SOME CLASSES OF STARLIKE AND CONVEX FUNCTIONS

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1. **Introduction :** Let S denote the class of functions of the form

$$(1.1)\quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are anatytic and univalent for $|z| < 1$. Then a necessary and sufficient condition for the function $w = f(z)$ to map $|z| < 1$ onto a star-shaped domain (i. e. f is starlike) is that

$$(1.2)\quad \mathrm{Re}\left\{\frac{z f'(z)}{f(z)}\right\} \geqslant o, \quad |z| < 1,$$

which implies that $|a_n| \leqslant n$; $n = 2, 3, \ldots$ Since the function $z / (1-z)^2 = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots$ maps $|z| < 1$ onto a star-shaped domain, the inequalities $|a_n| \leqslant n$ are sharp. We denote by S^* the class of analytic, univalent and starlike functions in the unit disk.

On the other hand, a necessary and sufficient condition for the function $w = f(z)$ to map $|z| < 1$ onto a convex domain is that

$$(1\ 3)\quad \mathrm{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \geqslant o, \quad |z| < 1,$$

which implies that $w = f(z)$ maps $|z| < 1$ onto a convex domain if and only if $z f'(z)$ maps $|z| < 1$ onto a star-shaped domain and furthermore $|a_n| \leqslant 1, n = 2, 3, \ldots$. Since the function $z / (1-z) = z + z^2 + z^3 + \ldots$ maps $|z| < 1$ onto the half-plane $\mathrm{Re}\{w\} > -\frac{1}{2}$, the inequalities $|a_n| \leqslant 1$ are sharp.

In the case of a convex function, the $\frac{1}{4}$ — theorem of Bieberbach for analytic and univalent function $w = f(z)$ can be improved as follows :

If the function $w = f(z)$ maps $|z| < 1$ onto a convex domain D, then D contains the disk $|w| < \frac{1}{2}$.

We denote by K the class of analytic, univalent and convex functions in the unit disk.

If we now consider a number α ($0 \leqslant \alpha < 1$) and denote by S^*_α the class of those functions $f(z) \in S^*$ such that

$$(1.4) \quad \operatorname{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \alpha, \quad |z| < 1,$$

then $f(z) \in S^*_\alpha$ is said to be starlike of order α.

A subclass $S^*_{[\alpha]}$ of S^*_α consisting of those f (z) for which

$$(1.5) \quad \left| z\frac{f'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad |z| < 1,$$

was considered by C. P. Mc Carty [1].

We can similarly define the class $K\alpha$ of analytic, univalent and convex functions of order α.

Let us now denote by S (n) the class of functions.

$$(1.6) \quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \geqslant 1),$$

that are analytic and univalent in the unit disk $|z| < 1$. Then f is said to be starlike of order α ($0 \leqslant \alpha < 1$), denoted by $f \in S^*_\alpha(n)$, if $\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} > \alpha$, ($|z| < 1$).

A Subclass $S^*_{[\alpha]}(n)$ of $S^*_\alpha(n)$ consists of those functions f (z) for which

$$(1.7) \quad \left| z\frac{f'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad |z| < 1.$$

Similarly f, defined by (1.6), is said to be convex of order α ($0 \leqslant \alpha < 1$), denoted by $f \in K\alpha(n)$ if

$$(1.8) \quad \operatorname{Re}\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \alpha, \quad |z| < 1.$$

One can similarly define $K_{[\alpha]}(n)$.

Let T be the subclass of functions, defined by (1 6), consisting of functions expressible of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} |a_k| z^k, \quad n \geqslant 1, \tag{1.9}$$

then $T^*_\alpha(n)$ and $C_\alpha(n)$ are defined respectively as the subclasses of T that are starlike of order α and convex of order α. One can similarly define $T^*_{[\alpha]}(n)$ and $C_{[\alpha]}(n)$. It may be noted that class $S^*_\alpha(1)$ is same as S^*_α. Recently M. S. Kasy [2] has considered the class $S^*_\alpha(2)$ where $a_3 = c$ is fixed and $|c| < 1$. Also H. Silverman [3] has considered the class $S^*_\alpha(1)$ and a subclass $T^*_\alpha(1)$ of $S^*_\alpha(1)$ where $f \in T^*_\alpha(1)$ is expressed in the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{1.10}$$

In a recent paper [4] the first author has considered some properties of the classes $S^*_\alpha(n)$, $S^*_{[\alpha]}(n)$ and $T^*_\alpha(n)$. Since the classes $S^*_\alpha(1)$ and $T^*_\alpha(1)$ are special cases of the classes $S^*_\alpha(n)$ and $T^*_\alpha(n)$ respectively, it is not appropriate [5] to derive any conclusion for $S^*_\alpha(n)$ or $T^*_\alpha(n)$ from that for $S^*_\alpha(1)$ or $T^*_\alpha(1)$ respectively by setting $a_2 = a_3 \dots = a_n = o$, because n is not fixed, but on the other hand n is a positive integral variable in the definition of $S^*_\alpha(n)$ or $T^*_\alpha(n)$. Furthermore any property like representation formula, distortion theorem, etc. of $S^*_\alpha(1)$ or $T^*_\alpha(1)$ comes out as a special case from that of $S^*_\alpha(n)$ or $T^*_\alpha(n)$ respectively.

In the present paper we shall discuss some properties of the above classes of starlike and convex functions.

2. Some properties of the classes S^*_α and K_α

Let $f \in S^*_\alpha$. Then we have

$$\operatorname{Re}\left\{\frac{zf'}{f}\right\} > \alpha. \quad |z| < 1$$

$$\text{or, } \operatorname{Re}\left\{\frac{zf'}{f} - \alpha\right\} > 0.$$

Now by virtue of the relation (33) of [6, P. 169] we obtain

$$(2.1) \quad \frac{1-r}{1+r} \leqslant \left|\frac{zf'}{f} - \alpha\right| \leqslant \frac{1+r}{1-r}, \quad |z| = r < 1.$$

In particular, when $\alpha = 0$ we obtain from (2 1)

$$(2.2) \quad \frac{1-r}{r(1+r)} \leqslant \left|\frac{f'}{f}\right| \leqslant \frac{1+r}{r(1-r)}. \quad |z| = r < 1,$$

which is a well-known [7, P 5] distortion theorem for univalent function f. Furthermore (2 1) implies that

$$(2\ 3) \quad \alpha - \frac{1+r}{1-r} \leqslant \operatorname{Re}\left\{\frac{zf'}{f}\right\} \leqslant \alpha + \frac{1+r}{1-r}$$

and

$$(2.4) \quad \alpha + \frac{1-r}{1+r} \leqslant \operatorname{Re}\left[\frac{zf'}{f}\right]; \quad \operatorname{Re}\left[\frac{zf'}{f}\right] \leqslant \alpha - \frac{1-r}{1+r}$$

Since $\operatorname{Re}\left\{\frac{zf'}{f}\right\} > \alpha$, we have from (2.3) and (2.4)

$$(2.5) \quad \alpha + \frac{1-r}{1+r} \leqslant \operatorname{Re}\left[\frac{zf'}{f}\right] \leqslant \alpha + \frac{1+r}{1-r}.$$

In other words, we have

$$(2.6)\quad \frac{1}{r}\left[\alpha+\frac{1-r}{1+r}\right]\leqslant\frac{\partial}{\partial r}\log|f(z)|\leqslant\frac{1}{r}\left[\alpha+\frac{1+r}{1-r}\right],$$

since $\operatorname{Re}\left(\frac{zf'}{f}\right)=r\frac{\partial}{\partial r}\log|f(z)|$.

Integrating (2 6) from r_1 to r_2, where $o<r_1<r_2<1$, we deduce

$$(2.7)\quad \frac{r_2^{\alpha+1}}{(1+r_2)^2}\Big/\frac{r_1^{\alpha+1}}{(1+r_1)^2}\leqslant\left|\frac{f(r_2 e^{i\theta})}{f(r_1 e^{i\theta})}\right|\leqslant\frac{r_2^{\alpha+1}}{(1-r_2)^2}\Big/\frac{r_1^{\alpha+1}}{(1-r_1)^2}$$

Making $\alpha=o$ and $r_1\to o$ in (2.7) we deduce a well-known [7] distortion theorem for univalent function f with $r=r_2$, viz.

$$(2.8)\quad \frac{r}{(1+r)^2}\leqslant|f(z)|\leqslant\frac{r}{(1-r)^2}.$$

It therefore follows from (2.2) and (2.8) that

$$(2.9)\quad \frac{1-r}{(1+r)^3}\leqslant|f'(z)|\leqslant\frac{1+r}{(1-r^{3}},$$

which is also well-known [7] for univalent function f.

Next let $f\in K_\alpha$. Then we have

$$\operatorname{Re}\left\{1+\frac{zf''}{f'}\right\}>\alpha,\quad |z|<1$$

$$\text{or}\ \operatorname{Re}\left\{\frac{zf''}{f'}+(1-\alpha)\right\}>o.$$

By virtue of the same relation (33) of [6, p. 169] we obtain

$$(2.10)\quad \frac{1-r}{1+r}\leqslant\left|\frac{zf''}{f'}+(1-\alpha)\right|\leqslant\frac{1+r}{1-r},\quad |z|=r<1.$$

Now (2.10) implies that

$$(2.11)\quad (\alpha-1)-\frac{1+r}{1-r}\leqslant\operatorname{Re}\left\{\frac{zf''}{f'}\right\}\leqslant(\alpha-1)+\frac{1+r}{1-r}$$

and

$$(2.12)\quad (\alpha-1 \;+\frac{1-r}{1+r} \leqslant \operatorname{Re}\left\{\frac{z\ ''}{f'}\right\};\ \operatorname{Re}\left\{\frac{z f''}{f'}\right\} \leqslant (\alpha-1)-\frac{1-r}{1+}$$

Since $\operatorname{Re}\left\{\frac{z f''}{f'}+1\right\} > \alpha$, we have from (2 11) and (2.12)

$$(2\ 13)\quad (\alpha-1)+\frac{1-r}{1+r} \leqslant \operatorname{Re}\left\{\frac{z f''}{f'}\right\} \leqslant (\alpha-1)+\frac{1+r}{1-r}$$

In other words, we have

$$(2.14)\quad \frac{1}{r}\left[(\alpha-1)+\frac{1-r}{1+r}\right] \leqslant \frac{\partial}{\partial r}\log|f'(z)| \leqslant \frac{1}{r}\left[(\alpha-1)+\frac{1+r}{1-r}\right],$$

since $\operatorname{Re}\left\{\frac{z f''}{f}\right\}=r\frac{\partial}{\partial r}\log|f'(z)|$.

Integrating (2.14) from o to r, we obtain

$$(2.15)\quad \frac{r^{\alpha}}{(1+r)^2} \leqslant |f'(z)| \leqslant \frac{r^{\alpha}}{(1-r)^2}$$

$$\text{Now } |f(z| = \left|\int_0^z f'(z)\,dz\right| \leqslant \int_0^r |f'(z)|\,dr \leqslant \int_0^{\alpha} \frac{r^{\alpha}}{(1-r)^2}$$

To obtain a lower bound for $|f(re^{i\theta})|$, we assume without any loss of generality, that $f(re^{i\theta}) = Re^{i\phi}$, where $R < \frac{1}{2}$, since otherwise there is nothing to prove It then follows that the straight line segment λ from o to $Re^{i\phi}$ lies entirely in the image of $|z| < 1$ by f (z). Hence λ corresponds to a path L in $|z| < 1$, which joins $z = o$ to $re^{i\theta}$. Thus we have

$$R = \int_{\lambda} |dw| = \int_L \left|\frac{dw}{dz}\right| |dz| \geqslant \int_0^r \frac{r^{\alpha}}{(1+r)^2}dr.$$

Hence we have

$$(2.16)\quad \int_0^r \frac{r^{\alpha}}{(1+r)^2}dr \leqslant |f(z)| < \int_0^r \frac{r^{\alpha}}{(i-r)^2}dr.$$

In particular, when $\alpha = o$, we obtain from (2.15) and (2.16)

$$(2.17)\quad \frac{1}{(1+r)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-r)^2}$$

and

$$(2.18)\quad \frac{r}{1+r} \leqslant |f(z)| \leqslant \frac{r}{1-r}$$

which are well-known [6, p. 225] more improved distortion theorems of convex univalent function f than the distortion theorems of univalent function f (or of starlike univalent function f by virtue of (2.8) and (2.9)).

Since we know that $K_o \equiv S^*_{\frac{1}{2}}$, we have (2.17) and (2.18) for any $f \in S^*_{\frac{1}{2}}$, which are well-known results of Zhuo—ren Wu [8].

Again $f \in K_o \Longleftrightarrow z f' \in S^*_o$, so that (2.17) follows at once from (2.8).

Similarly $f \in K_\alpha \Longleftrightarrow z f' \in S^*_\alpha$, so that (2.14) follows at once from (2.6) and consequently (2.15) and (2.16) can be rediscovered.

3. Some properties of the classes $S^*_{[\alpha]}(n)$ and $K_{[\alpha]}(n)$

It may be noted that Ming—Po Chen [9] considered the class $\mathscr{A}$ of functions expressible of the form :

$$(3.1)\quad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,\ n \geqslant 1,$$

satisfying the condition

$$(3.2)\quad \left| \frac{z f'(z)}{f(z)} - 1 \right| < \alpha'$$

for a given α, $o < \alpha \leqslant 1$, for $|z| \leqslant 1$. He remarked that every function of the above class $\mathscr{A}$ is a starlike function (of order zero), because in the introduction he only introduced starlike functions defined by the necessary and sufficient condition

Re $\left\{\frac{z f'}{f}\right\} > 0$ for $|z| < 1$, but actually every function $\in \mathscr{A}$ must be a starlike function of order $(1-\alpha)$. He derived the following properties for the class $\mathscr{A}$.

Property 1 (Representation theorem)

$f(z) \in \mathscr{A}$ if and only if

$$f(z) = z \exp\left(\int_0^z t^{n-1} \phi(t)\, dt\right), \tag{3 3}$$

where $\phi(z)$ is analytic and satisfies the condition

$|\phi(z)| \leqslant \alpha$, $0 < \alpha \leqslant 1$, for $|z| < 1$.

Property 2 (Distortion theorems)

For all $f(z) \in \mathscr{A}$ the following properties hold :

$$|z| \exp\{-\alpha(|z|^n)/_n\} \leqslant |f(z)| \leqslant |z| \exp\{\alpha(|z|^n)/_n\} \tag{3.4}$$

and

$$(1-\alpha|z|^n) \exp\{-\alpha(|z|^n)/_n\} \leqslant |f'(z)| \leqslant (1+\alpha|z|^n) \exp\{\alpha(|z|^n)/_n\}. \tag{3.5}$$

Now for the class $S^*_{[\alpha]}(n)$, we are to change α to $(1-\alpha)$ in the discussion of Ming-Po Chen, so that $\alpha \in [0, 1)$ and we obtain

Property 1′ $f(z) \in S^*_{[\alpha]}(n)$ if and only if

$$f(z) = z \exp\left[\int_0^z t^{n-1} \phi(t)\, dt\right], \tag{3.6}$$

where $\phi(z)$ is analytic and satisfies the condition

$|\phi(z)| \leqslant 1-\alpha$, $\alpha \in [0, 1)$, $|z| < 1$.

Property 2′ If $f(z) \in S^*_{[\alpha]}(n)$, then

$$|z| \exp\{(\alpha-1)(|z|^n)/_n\} \leqslant |f(z)| \leqslant |z| \exp\{(1-\alpha)(|z|^n)/_n\} \tag{3.7}$$

and

$$(1+(\alpha-1)|z|^n) \exp\{(\alpha-1)(|z|^n)/_n\} \leqslant |f'(z)| \leqslant (1+(1-\alpha)|z|^n) \exp\{(1-\alpha)(|z|^n)/_n\}. \tag{3.8}$$

The proof of Property 2′ can be shortened (unlike Ming-Po Chen) as follows :

Since
$$\left| \int_0^z t^{n-1} \phi(t)\, dt \right|$$
$$\leqslant \int_0^{|z|} |t|^{n-1} |\phi(t)| \, |dt|$$
$$\leqslant \int_0^{|z|} |t|^{n-1} (1-\alpha) |dt| = (1-\alpha)(|z|^n)/n,$$

(3.7) follows by virtue of Property 1′

Again (3.8) follows by applying the triangle inequality to $f'(z) = (1+z^n \phi(z)) f(z) / z$ and by (3. 7).

We now remark that our representation for $f \in S^*_{[\alpha]}(n)$ yields the following representation of Mc Carty for $S^*_{[\alpha]}(1)$ as a particular case

$$(3.9) \quad f \in S^*_{[\alpha]}(1) = z \exp \{ \int_0^z \phi(t)\, dt \},$$

where $\phi(z)$ is analytic for $|z| < 1$ and $|\phi(z)| < 1-\alpha$ for $|z| < 1$ and $\alpha \in [0, 1)$, which dearly shows that it is not possible to derive the representation of $S^*_{[\alpha]}(n)$ from that of $S^*_{[\alpha]}(1)$.

Again our distortion theorems for $f \in S^*_{[\alpha]}(n)$ yields the following distortion theorems of Mc Carty for $S^*_{[\alpha]}(1)$ as a particular case :

If $f(z) \in S^*_{[\alpha]}(1)$, then

$$(3.10) \quad |z| e^{(\alpha-1)|z|} \leqslant |f(z)| \leqslant |z| e^{(1-\alpha)|z|}$$

and

$$(3.11) \quad (1 + (\alpha-1)|z|) e^{(\alpha-1)|z|} \leqslant |f'(z)|$$
$$\leqslant (1 + (1-\alpha)|z|) e^{(1-\alpha)|z|},$$

which again reveals the fact that it is not possible to derive distortion theorems of $S^*_{[\alpha]}(n)$ from those of $S^*_{[\alpha]}(1)$.

Next we notice that $f \in K_{[\alpha]}(n) \Longleftrightarrow z f' \in S^*_{[\alpha]}(n)$,

so that we obtain from Property 1′ and Property 2′ the following properties :

Property 3 $f(z) \in K_{[\alpha]}(n)$ if and only if

$$(3.12)\quad f'(z) = \exp\left\{\int_0^z t^{n-1}\phi(t)\,dt\right\},$$

where $\phi(z)$ is analytic and satisfies the condition

$$|\phi(z)| \leqslant 1-\alpha,\ \alpha \in [0, 1),\ |z| < 1$$

Property 4 If $f(z) \in K_{[\alpha]}(n)$, then

$$(3.13)\quad \exp\{(\alpha-1)(|z|^n)/_n\} \leqslant |f'(z)| \leqslant \exp\{(1-\alpha)(|z|^n)/_n\}.$$

4. *Some properties of the classes $T^*_\alpha(n)$ and $C_\alpha(n)$*

Let $f \in T^*_\alpha(n)$. Then from the result of [5, p 117) we have the following distortion.

$$(4.1)\quad |z| - \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1} \leqslant |f(z)| \leqslant |z| + \left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^{n+1}$$

Now we notice that $f \in C_\alpha(n) \Longleftrightarrow z f' \in T^*_\alpha(n)$, so that we at once obtain from (4.1)

$$(4.2)\quad 1-\left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^n \leqslant |f'(z)| \leqslant 1+\left(\frac{1-\alpha}{n+1-\alpha}\right)|z|^n,$$

which is (2.12) in the work [5, p. 119].

The above method of deriving (4.2) indicates that (2 12) of [5, p 119] need not be derived independently and moreover Lemma 2 of the first author of the present paper, which is utilized in deriving (2.12) of [5, p.119], follows immediately from Lemma 1 of the first author of the present paper by using the fact $f \in C_\alpha(n) \Longleftrightarrow z f' \in T^*_\alpha(n)$, which was already pointed out by the first author of the present paper in [4].

REFERENCES

[1] C. P Mc Carty—Proc. Amer-Math. Soc. 43 (1974), 361-366

[2] M. S. Kay—The Math. Education 18 (1984), 156—159

[3] H. Silverman—Proc. Amer. Math. Soc. 51 (1975), 109—116

[4] S. K. Chatterjea—Jour. Pure Math. 1 (1981), 23—26

[5] H M. Srivastava, Shigeyoshi Owa and S.K. Chatterjea—Rend. Sem. Mat. Univ. Padova, LXXVII (1987), 115—124

[6] Z. Nehari—Conformal Mapping, Dover Publ. Inc. New York (1975)

[7] W. K. Hayman—Multivalent Functions, Combridge Univ. Press (1967).

[8] Zhuo-ren Wu—Acta Math. Sinica 6 (1956), 476-489

[9] Ming—Po Chen—Nanta Math 8 (1975), 79—82

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ON SOME CLASSES OF UNIVALENT FUNCTIONS

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1. G. S. Salagean [4] defined the classes $S_n(\alpha)$ of univalent and normalized functions f in the unit disc $U=\{z\epsilon C;\ |z|<1\}$ by

$$S_n(\alpha)=\left\{f: \mathrm{Re}\ \frac{D^{n+1}f(z)}{D^nf(z)}>\alpha,\ z\epsilon U\right\},$$

where $n\epsilon N_o=N\cup\{o\}$ and $\alpha\epsilon[o, 1)$,

by means of the following differential operators :
$D^of(z)=f(z)$, $D^1f(z)\equiv Df(z)=zf'(z)$, $D^nf(z)=D(D^{n-1}f(z,))$.

It may be of interest to observe that his $S_o(\alpha)$ is the well known class S^*_α of starlike functions of order α and his $S_1(\alpha)$ is the well known class K_α of convex functions of order α. Furthermore $S_1(\alpha)\subset S_0(\alpha)$ by virtue of his result $S_{n+1}(\alpha)\subset S_n(\alpha)$. Now for Salagean classes $S_n(\alpha)$ one can deduce that (cf. [1])

$$(1.1)\quad \alpha+\frac{1-r}{1+r}\leqslant \mathrm{Re}\ \frac{D^{n+1}f(z)}{D^nf(z)}\leqslant \alpha+\frac{1+r}{1-r},\quad |z|=r<1.$$

since $\dfrac{D^{n+1}f(z)}{D^nf(z)}=\dfrac{D(D^nf(z)}{D^nf(z)}=\dfrac{z\,(D^nf(z))'}{D^nf(z)}$,

$$\mathrm{Re}\ \left\{\frac{z\,(D^nf(z))'}{D^nf(z)}\right\}=r\frac{\partial}{\partial r}\ \log|D^nf(z)|.$$

It therefore follows from (1.1) that

$$(1.2]\quad \frac{1}{r}\left(\alpha+\frac{1-r}{1+r}\right)\leqslant\frac{\partial}{\partial r}\log|D^n f(z)|\leqslant\frac{1}{r}\left(\alpha+\frac{1+r}{1-r}\right).$$

In particular, when $r=o$, one obtains from (1.2)

$$(1{,}3)\quad \frac{1}{r}\left(\alpha+\frac{1-r}{1+r}\right)\leqslant\frac{\partial}{\partial r}\log|f(z)|\leqslant\frac{1}{r}\left(\alpha+\frac{1+r}{1-r}\right).$$

which was obtained by us in a previous work [1].

Again, when $n=1$, one obtains from (1.2)

$$(1.4)\quad \frac{1}{r}\left\{(\alpha-1)+\frac{1-r}{1+r}\right\}\leqslant\frac{\partial}{\partial r}\log|f'(z)|\leqslant\frac{1}{r}\left\{(\alpha-1)+\frac{1+r}{1-r}\right\}$$

Integrating (1.4) from o to r, one obtains

$$(1.5)\quad \frac{r^{\alpha}}{(1+r)^2}\leqslant|f'(z)|\leqslant\frac{r^{\alpha}}{(1+r)^2},$$

from which one can derive

$$(1.6)\quad \int_o^r\frac{r^{\alpha}}{(1+r)^2}\,dr\leqslant|f(z)|\leqslant\int_o^r\frac{r^{\alpha}}{(1-r)^2}\,dr,$$

which was also obtained by us in the previous work [1].

Since Salagean remarked that all functions in $S_n(\alpha)$, $n\in N_o$, $\alpha\in[o,1)$ are starlike and all functions in $S_n(\alpha)$, $n\in N$, $\alpha\in[o,1)$ are convex, we can remark that the same distortion theorem (1 2) is true for starlike functions and convex functions according as $n\in N_0$ and $n\in N$. It is evident that $f\in S_n\ \alpha)$, $n\in N\Rightarrow zf'\in S_n(\alpha)$, $n\in N_0$.

Next we shall show that the co-efficient result in connection with $S_n(\alpha)$ in theorem 4 of Salagean follows at once from that in connection with S^*_{α} due to H. Silverman and E. M. Silvia [4]. In fact, we notice that

$$S_n(\alpha)=\left\{f:\operatorname{Re}\frac{z\phi'(z)}{\phi(z)}>\alpha,\ z\in U\right\},$$

where $\phi(z)\equiv D^n f(z)$.

Now $f(z)=z+\sum_{j=2}^{\infty} a_j z^j$ and therefore

$$\phi(z) \equiv D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \in S^*_{\alpha},\ n \in N_0.$$

Let us put $\Phi=z+\sum_{j=2}^{\infty} b_j z^j$, where $b_j=j^n a_j$ and $b_2=2^n a_2$ ($|a_2|=a$). Then from the coefficient result of Silverman and Silvia we get

$$|b_j| \leqslant \frac{1+|b_2|}{3-2\alpha}\ \frac{\prod_{k=2}^{j}(k-2\alpha)}{(j-1)!},\quad j=3, 4, \ldots$$

$$\text{i.e. } |a_j| \leqslant \frac{1+2^n a}{3-2\alpha}\ \frac{\prod_{k=2}^{j}(k-2\alpha)}{(j-1)!\,j^n}$$

which is the coefficient result derived by Salagean who offered a proof in about two pages. In fact, putting $n=o$ and $n=1$ successively in the above coefficient result of Silverman and Silvia for $S^*(\alpha)$ and K_{α} respectively.

2. S Ruscheweyh [3] defined another classes K_n of univalent and normalized function f in the unit disc $U=\{z \in C\ ;\ |z|<1\}$ by

$$K_n=\left\{f\ ;\ \mathrm{Re}\ \frac{\mathfrak{D}^{n+1} f(z)}{\mathfrak{D}^{n} f(z)} > \tfrac{1}{2},\ z \in U\right\},$$

where $n \in N_0=N \cup \{o\}$ and $\mathfrak{D}^n f = z\,(z^{n-1} f)^{(n)} / n\,!$.
It may be of interest to observe that his K_0 is the well known class $S_{\frac{1}{2}}^*$ of starlike functions of order $\frac{1}{2}$ and his K_1 is the well known class K of convex functions of order zero.
Thus $K_1 \equiv K \subset K_0$ is a special case of the following result of Ruscheweyh

$$K_{n+1} \subset K_n,\ n \in N_0$$

Now for Ruscheweyh classes K_n one can deduce that (cf. [1])

$$(2.1) \qquad \tfrac{1}{2}+\frac{1-r}{1+r} \leqslant \operatorname{Re} \frac{\mathfrak{D}^{n+1} f}{\mathfrak{D}^{n} f} \leqslant \frac{1}{2}+\frac{1+r}{1-r}, \quad |z| = r<1$$

Since we know that [2]

$$\frac{\mathfrak{D}^{n+1} f}{\mathfrak{D}^{n} f} = \frac{1}{n+1}\left(z\frac{(\mathfrak{D}^{n} f)'}{\mathfrak{D}^{n} f} + n \right),$$

we have

$$\operatorname{Re} \frac{\mathfrak{D}^{n+1} f}{\mathfrak{D}^{n} f} = \frac{1}{n+1}\left(n + r\frac{\partial}{\partial r} \log | \mathfrak{D}^{n} f | \right).$$

It therefore follows from (2.1) that

$$(2.2) \qquad \tfrac{1}{2}+\frac{1-r}{1+r} \leqslant \frac{1}{n+1}\left(n + r\frac{\partial}{\partial r} \log | \mathfrak{D}^{n} f | \right) \leqslant \tfrac{1}{2}+\frac{1+r}{1-r}.$$

In particular, when $n=o$, obtains from (2.2)

$$(2.3) \qquad \frac{1}{r}\left(\tfrac{1}{2}+\frac{1-r}{1+r}\right) \leqslant \frac{\partial}{\partial r} \log | f(z) | \leqslant \frac{1}{r}\left(\tfrac{1}{2}+\frac{1+r}{1-r}\right),$$

which is a particular case of our previous result [1].

Again, when $n=1$, one obtains from (2.2)

$$(2.4) \qquad \frac{1}{r}\left(-\tfrac{1}{2}+\frac{1-r}{1+r}\right) \leqslant \frac{\partial}{\partial r} \log | f'(z) | \leqslant \frac{1}{r}\left(-\tfrac{1}{2}+\frac{1+r}{1-r}\right).$$

Integrating (2 4) from o to r one obrains

$$(2.5) \qquad \frac{\sqrt{r}}{(1+r)^2} \leqslant | f'(z) | \leqslant \frac{\sqrt{r}}{(1-r)^2},$$

from which one can derive

$$(2.6) \qquad \int_0^r \frac{\sqrt{r}}{(1+r)^2}\,dr \leqslant | f(z) | \leqslant \int_0^r \frac{\sqrt{r}}{(1-r)^2}\,dr$$

i. e. $\tan^{-1}\sqrt{r}-\frac{\sqrt{r}}{1+r} \leqslant | f(z) | \leqslant \frac{\sqrt{r}}{1-r} - \tfrac{1}{2}\log\frac{1+\sqrt{r}}{1-\sqrt{r}}$.

It may be noted that (2.5) and (2.6) are well compared with the following known results for convex functions of order zero

$$(2.7)\quad \frac{1}{(1+r)^2} \leqslant | f'(z) | \leqslant \frac{1}{(1-r)^2}$$

$$(2.8)\quad \frac{r}{1+r} \leqslant | f(z) | \leqslant \frac{r}{1-r}$$

Like §1 we can remark that all functions in K_n, $n \epsilon N_o$, are starlike of order $\frac{1}{2}$ and all functions in K_n, $n \epsilon N$, are convex of order zero and therefore the same distortion theorem (2.2) is true for starlike functions of order $\frac{1}{2}$ and convex functions of order zero according as $n \epsilon N_0$ and $n \epsilon N$.

Finally we shall point out the geometrical structure of the elements in K_n. To this end, we notice that

$$K_n = \{ f : \mathrm{Re} \left[\frac{1}{n+1} (n + \frac{z(\mathfrak{D}^n f)'}{\mathfrak{D}^n f)} \right] > \tfrac{1}{2},\ z \epsilon U \}$$

$$\text{i. e. } K_n = \{ f : \mathrm{Re} \frac{z\Phi'(z)}{\Phi(z)} > \frac{1-n}{2},\ z \epsilon U \},$$

where $\Phi = \mathfrak{D}^n f$.

Now $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and therefore

$$\Phi(z) \equiv \mathfrak{D}^n f(z) = [z (1-z)^{-n-1}] * f(z)$$

$$= z + \sum_{j=2}^{\infty} \frac{(n+1)_{j-1}}{(j-1)!} a_j z^j \epsilon \overset{*}{S}_{(1-n)/2}, \quad n \epsilon N_o,$$

although $S^*_{(1-n)/2}$ is not known in the literature except for $n = o$ and $n = 1$. Indeed, when $n = o$. $\Phi \equiv f \epsilon S^*_{\frac{1}{2}}$ and when $n = 1$, $\Phi \epsilon S^*_0$ or in other words $f \epsilon K$. Yet Ruscheweyh defined the class K_{-1} as the set of functions f with Re $f(z)/z > \frac{1}{2}$, $z \epsilon U$. Actually when $n = -1$, $\Phi \epsilon S_1{}^*$, which is also not known in the literature. So the notation $\overset{*}{S}_{(n-1)/2}$ is not unnatural in the geometric function theory—which furnishes an open problem in this paper. When n is replaced by an arbitrary real number $\alpha \geqslant -1$, it may be of interest to observe that $\Phi(z) \equiv \mathfrak{D}^{\alpha} f(z) \in S^*_{(1-\alpha)/2}$.

REFERENCES

[1] Chatterjea, S. K. and Adhikary Abir Kumar—On some classes of starlike and convex functions, this Journal vol 6 (1987-88), 63.

[2] Fukui, S. and Sakaguchi, K.—An extension of a theorem of S. Ruscheweyh, Bull. Fac. Edu. Wakayama Univ. Nat. Sci. 29 (1980), 1—

[3] Ruscheweyh, Stephen—New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975); 109—115

[4] Salagean, Grigore Stefan—Subclasses of univalent functions, Lecture Notes in Math. 1013 (Springer—Verlag (1983)), 362--372.

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SINGULARITIES OF LIFE

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Life is a function of environment and incidents rather than a positive and independent existence Although V.A. Kostitzin [1] pointed out the stabilizing effect of living matter with respect to sudden variations in the composition of the atmosphere, yet we like to point out that there are certain states of environment and incidents which retard the smoothness of life, rather the growth (dm/dt) of life, and these we term as singularities of life and so we may observe that life function $f(z)$ is not analytic (cf. Life is not a bed of roses) and this has resulted for it in a very special form of the struggle for existence. Each living organism is usually represented by a point in a multidimensional space which symbolically represents many vital factors and here we are considering only singularities of life and for this reason we like to represent life simply as $f(z)$, which is not a Taylor—expansible function. Indeed, singularities z_i are essential in life for combustion process. Like the flame life tries to consume singularities and then it may grow or may not grow. So, although singularities prevent the growth of life and the smooth movement of life for the time being, yet they help life in the struggle for existence and survival of the fittest, just like the fire which sends out sparks, of which a few die, but many, falling upon favourable situation, flare up as a second generation. The singularities which life consumes and then grows, may be termed as polar, whereas the singularities which life fails to consume wholly and then does not grow, may be termed as essential So an essential singularity of life causes death. Symbolically we may write as follows:

$$\left[\begin{array}{c}\text{Life affected}\\ \text{by Singularities}\end{array}\right]\begin{array}{l}\nearrow\ (\text{ recovery from Singularities })\\[2ex] \searrow\ (\text{ death })\end{array}$$

Now death has also a great singificance in life function—it teaches us how to control these types of essential singularities in future through new discoveries. So death can be compared with the policy of resignation or sacrifice. From historical precedent we observe that Geometers have had to resign themselves to the fact that Euclid's parallel axiom cannot be proved and as a reward of this sacrifice came the new geometries of Bolyai, Lobatchewski and Riemann.

A model in this connection will be considered in a forthcoming publication.

REFERENCE

[1] Lecture Notes in Biomathematics 22—Springer-Verlag (1978), pp. 443-484

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QUASI INNER PRODUCT FOR SOME BILINEAR GENERATING FUNCTIONS OF LEGENDRE POLYNOMIALS

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We recall the method of quasi inner product, introduced by L.R. Bragg [1], which is very convenient for developing results for some bilinear generating functions of Legendre polynomials.

First we observe that

$$(1) \quad \sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2tx+t^2)^{-\frac{1}{2}} \equiv f(x, t)$$

and

$$(2) \quad \sum_{n=0}^{\infty} P_n(y)\, t^n = (1-2ty+t^2)^{-\frac{1}{2}} \equiv f(y, t),$$

where $|t| < 1$.

Then by Bragg's method we have

$$f(x,t) \circ f(y,t) = \sum_{n=0}^{\infty} P_n(x)\, P_n(y)\, t^{2n}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (1-2xte^{i\theta}+t^2e^{2i\theta})^{-\frac{1}{2}} (1-2yte^{-i\theta}+t^2e^{-2i\theta})^{-\frac{1}{2}} d\theta.$$

$$(3) \quad = \frac{1}{2\pi} \int_0^{2\pi} [1+t^4+4t^2xy+2t^2\cos 2\theta-2t(1+t^2)(x+y)\cos\theta - 2it(1-t^2)(x-y)\sin\theta]^{-\frac{1}{2}} d\theta,$$

which may be compared with the results of Legendre [2], Watson [3], Maximon [4] and Carlitz [5] for the generating series

$$\sum_{n=0}^{\infty} P_n(x) P_n(y) t^n.$$

In particular, we have from (3) on using $y=x$

$$\sum_{n=0}^{\infty} [P_n(x)]^2 t^{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} [1 + t^4 + 4t^2x^2 + 2t^2\cos 2\theta - 4xt(1+t^2)\cos\theta]^{-\frac{1}{2}} d\theta,$$

which is a special case of Bragg's result [1, formula (4.7)].

Again using $y=1$ in (3) and comparing it with (1) we get the following integral evaluation

$$(5) \quad \frac{1}{2\pi}\int_0^{2\pi} (1 - te^{-i\theta})^{-1} (1 - 2xte^{i\theta} + t^2e^{2i\theta})^{-\frac{1}{2}} d\theta$$

$$= (1 - 2t^2x + t^4)^{-\frac{1}{2}},$$

which includes the result (when $x=1$) :

$$(6) \quad \int_0^{2\pi} \frac{d\theta}{1 - 2t\cos\theta + t^2} = \frac{2\pi}{1 - t^2},$$

which also follows from (4) on using $x=1$.

Again if we observe that

$$(7) \quad \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) P_n(x) t^n = \tfrac{1}{2}(1 - t^2)(1 - 2tx + t^2)^{-\frac{3}{2}}$$

and

$$(8) \quad \sum_{n=0}^{\infty} P_n(y) t^n = (1 - 2ty + t^2)^{-\frac{1}{2}},$$

where $| t < 1$,

then by Bragg's method we have

$$(9) \quad \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) P_n(x) P_n(y) t^{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1 - t^2e^{2i\theta})(1 - 2tye^{-i\theta} + t^2e^{-2i\theta})}{2\,[(1 - 2txe^{i\theta} + t^2e^{2i\theta})(1 - 2ty\,e^{-i\theta} + t^2e^{-2i\theta})]^{\frac{3}{2}}} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-t^4-2it^2\sin 2\theta+2t^3y\,(\cos\theta+\sin\theta)-2ty\,(\cos\theta-i\sin\theta)\;d\theta}{2\left[\,1-2t\,(xe^{i\theta}+ye^{-i\theta})+2t^2\cos 2\theta-2t^3\,(xe^{-i\theta}+ye^{i\theta})+4t^2xy+t^4\,\right]^{\frac{3}{2}}},$$

which may be compared with the result of Watson [3].

In particular, we have from (9) on using y=x

$$(10)\quad \sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, P^{2n}(x)\; t^{2n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-t^2)\,(1+t^2-2tx\cos\theta)\,d\theta}{2\left[\,(1+t^2-2tx\cos\theta)^2-4t^2\,(1-x^2)\sin^2\theta\,\right]^{\frac{3}{2}}}$$

Again using y= 1 in (9) and comparing it with (7) we obtain

$$(11)\quad \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-t^2e^{2i\theta})\;d\theta}{(1-te^{-i\theta})\,(1-2txe^{i\theta}+t^2e^{2i\theta})}$$

$$=(1-t^4)\,(1-2t^2x+t^4)^{-\frac{3}{2}},$$

which includes the result (when x=1)

$$(1.2)\quad \int_0^{2\pi} \frac{d\theta}{(1-2t\cos\theta+t^2)^2} = \frac{2\pi\,(1+t^2)}{(1-t^2)^3}$$

which also follows from (10) on using x=1, by virtue of the fact

$$\sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, t^{2n} = \tfrac{1}{2}\,\frac{(1+t^2)}{(1-t^2)^2}\,.$$

REFERENCES

[1] L. R. Bragg—SIAM J. MATH. ANAL. 17 (1986), 220-230

[2] E. T. Whittaker & G N. Waston—A course of modern analysis, Combridge Univ. Press (1952), p. 335

[3] G. N. Watson—J. Lond. Math. Soc. 8 (1933), 289-292

[4] G. Maximon—Norske Videnskabers Selskab Forkandlinger 20 (1957), 82-86

(5) L. Carlitz—Boll. Un. Mat. Ital (3), XIV (1959), 6-9

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GENERAL FIXED POINT PRINCIPLE

ASIT KUMAR SARKAR

The general fixed point principle is based on the following two simple conditions :

(H1) : The operator $T : \overline{G} \to X$ is compact on the closure of G, where G is a non-empty open bounded subset of the real Banach space X

and

(H2) : The image set $T(\partial G)$ contains no points of exterior rays i.e., there is an elmeent $x_0 \in G$ such that
$T(x) \neq x_0 + \beta(x - x_0)$ for all $x \in \partial G$ and $\beta > 1$.

In 1986, E. Zeidler [1] has proved the following theorem by the application of fixed point index :

THEOREM : 1 : (Zeidler [1]) : If (H1) and (H2) be satisfied, then T has a fixed point in G.

The importance of the condition (H2) lies in the fact that either of the following well-known conditions implies (H2) :

(a) *Rothe condition :* The set G is convex and $T(\partial G) \subseteq \overline{G}$.

(b) *Leray-Schauder condition :* $0 \in G$ and $tTx \neq x$ for all $(x, t) \in \partial G \times (0,1)$.

(c) *Altman condition :* There is an $x_0 \in G$ such that

$$\| Tx - x \|^2 \geqslant \| Tx - x_0 \|^2 - \| x - x_0 \|^2 \text{ for all } x \in \partial G.$$

REMARK : Let (H1) be given The condition (H2) will be satisfied and thus T will have a fixed point in G if the followiag condition is satisfied :

(H3) Let $0 \in G$ and $\| x - T(x) \| \geqslant \| T(x) \|$ for all $x \in \partial G$.

The object of this paper is to prove the following fixed point theorem by using the conditions (H1) and (H3) :

THEOREM : 2 If (H1) and (H3) be satisfied, then T will have a fixed point in G.

Proof of Theorem : 2 We shall merely show that the condition (H2) will be satisfied if (H3) holds.

Indeed, we take $x_0 = 0 \in G$, then (H2) becomes $T(x) \neq \beta x$ for all $x \in \partial G$ and $\beta > 1$.

If possible let, $T(x) = \beta x$ with $x \in \partial G$ and β is a real number.

Then from (H3) we get

$$\| x - T(x) \| \geqslant \| T(x) \| \text{ for all } x \in \partial G$$

$$\Rightarrow \quad \| x - \beta x \| \geqslant \| \beta x \|$$

$$\Rightarrow \quad \| x - \beta x \|^2 \geqslant \| \beta x \|^2$$

$$\Rightarrow \quad (1 - \beta)^2 \| x \|^2 \geqslant \beta^2 \| x \|^2$$

$$\Rightarrow \quad (1 - \beta)^2 \geqslant \beta^2$$

$$\Rightarrow \quad \beta \leqq \tfrac{1}{2} \text{, which contradicts (H2).}$$

Hence $T(x) \neq \beta x$ for all $x \in \partial G$ and $\beta > 1$ i.e., we can say that (H2) will be satisfied if (H3) holds and consequently T will have a fixed point in G.

REFERENCE

[1] *E. Zeidler :* Non-linear Functional Analysis & Applications : Fixed Point Theorems I; (1986) Springer -Verlag, New York.

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A NOTE ON BURCHNALL'S OPERATIONAL FORMULA FOR HERMITE POLYNOMIALS

S. P. CHAKRABARTY

In 1941 J. L. Burchnall [1] obtained the following remarkable operational formula for the Hermite polynomials

$$(1)\quad (2x-D)^n y = \sum_{r=o}^{n} (-1)^r \binom{n}{r} H_{n-r}(x)\, D^r\, y.$$

Later, in 1967, R. M. Wilcox [2] obtained the following general operational formula

$$(2)\quad (P+Q)^n = \sum_{s=o}^{[n/2]} \sum_{k=o}^{n-2s} (\tfrac{1}{2}c)^s\, n!Q^k P^{n-2s-k} \{s!\,k!\,(n-2s-k)!\}^{-1}$$

where the commutator of operators P and Q denoted by [P, Q] satisfies

$[P, Q] \equiv PQ - QP = cI$, c being a number and I the unit operator.

Now if we use $P = -D$, and $Q = 2x$ then $[P, Q] = -2I$ and therefore by (2) we have

$$(3)\quad (2x-D)^n y =$$

$$\sum_{s=o}^{[n/2]} \sum_{m=o}^{n-2s} (-1)^s\, n!\{s!\,m!\,(n-2s-m)!\}^{-1} (2x)^m (-D)^{n-2s-m} y.$$

Now we shall show the equivalence of the results (1) and (3). Indeed if we use the series manipulations viz.,

$$(4) \quad \sum_{m=0}^{n} \sum_{s=0}^{[m/2]} a_m b_s = \sum_{s=0}^{[n/2]} \sum_{m=0}^{n-2s} a_{m+2s} b_s$$

we get from (3)

$$(2x-D)^n y = \sum_{m=0}^{n} \sum_{s=0}^{[m/2]} (-1)^{n-m} \binom{n}{m} (-1)^s m! \{(m-2s)!\, s!\}^{-1} (2x)^{m-2s} D^{n-m} y$$

$$= \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} H_m(x) D^{n-m} y$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} H_{n-r}(x) D^r y \text{ (using } n-m = r \text{)},$$

which is exactly Burchnall's formula (1).

Next we consider the formula

$$(5) \quad H_{n+m}(x) = m!\, n! \sum_{r=0}^{\min(m,\, n)} \frac{(-2)^r}{(m-r)!\,(n-r)!\,r!} H_{m-r}(x)\, H_{n-r}(x).$$

Although Burchnall proved the formula (5) from (1), we like to prove (5) directly by employing Wilcox formula (2). In fact we notice that $H_{m+n}(x) = (2x-D)^n H_m(x)$

$$= \sum_{s=0}^{[n/2]} \sum_{k=0}^{n-2s} (-1)^s n! \{s!\, k!\, (n-2s-k)!\}^{-1} (2x)^k (-D)^{n-2s-k} H_m(x)$$

$$\text{i.e.} = \sum_{s=0}^{[n/2]} \sum_{k=0}^{n-2s} n!\, 2^{n-2s} (-1)^{n-2s-k} \binom{m}{m-2s-k} x^k H_{m-n+2s+k}(x), \; n \geq m.$$

Again if we use the series manipulation (4) we obtain

$$H_{m+n}(x) = n! \sum_{k=0}^{n} \sum_{s=0}^{[k/2]} \frac{2^{n-2s}(-1)^{n+s-k}}{s!\,(k-2s)!} \binom{m}{n-k} x^{k-2s} H_{m-n+k}(x)$$

$$= n! \sum_{k=0}^{n} \frac{H_{m-n+k}(x)}{k!} (-2)^{n-k} \binom{m}{n-k} H_k(x).$$

Now using $n-k = r$ we get

$$H_{m+n}(x) = \sum_{r=0}^{n} (-2)^r \binom{m}{r} \binom{n}{r} H_{m-r}(x)\, H_{n-r}(x) \quad \text{when } n \leqslant m'$$

which implies (5).

I am indebted to Prof. S. K. Chatterjea of Calcutta University, Department of Pure Mathematics for his kind help in the preparation of this work.

REFERENCES

[1] J. L. Burchnall—Quart. J. Math., (Oxford) 12 (1941), 9-11

[2] Wilcox, R. M., : J, Math. Phys, 8, (1967) 962.

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Dedicated to the memory of

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ON WEAK BI-IDEAL OF A SEMIGROUP

M. K. SEN and T. K. DUTTA

Let S be a semigroup. A nonempty subset A of S is said to be a weak bi-ideal of S (written in short w. bi-ideal) if $aSa \subseteq A$, $\forall\, a \in A$. In § 1 of this paper we have studied some properties of w. bi-ideal of a semigroup. In § 2 some characterizations of normal semigroup, regular semigroup, left regular semigroup and right regular semigroup, completely regular semigroup and homogroup have been proved in terms of w. bi-ideals. In § 3 we have defined prime w. bi-ideal and semiprime w. bi-ideal and studied them.

§ 1. We begin with the following definition.

Definition. A subsemigroup A of a semigroup S is called a bi-ideal of S if
$ASA \subseteq A$ [4].

Proposition 1.1. Every bi-ideal of a semigroup S is a w. bi-ideal; hence every one sided ideal is a w. bi-ideal of S.

Proof. The proof is obvious.

Remark. The converse of the above result is not true. This follows from the following example.

Example. Let $S=\{ (a, b) \mid a, b \in Z,$ the ring of integers $\}$. S is a semigroup w.r.t. the multiplication defined as follows: $(a, b)\,(c, d)=(a, d)$. Let $A=\{ (a, a) \mid (a, a) \in S \}$. Then A is a w. bi-ideal of S, since $(a, a)\,(x, y)\,(a, a)=(a, a) \in A$, $\forall\, (x, y) \in S$. It can be easily verified that A is neither a bi-ideal nor an one-sided ideal of S.

We now state some elementary properties of w. bi-ideals in a semigroup S.

1. The intersection of two w. bi-ideals is a w. bi-ideal if it is not empty.
2. The join of two w. bi-ideals is a w. bi-ideal.
3. If A and B are two w. bi-ideals then AB is also a w. bi-ideal.

Proposition 1.2. A semigroup S will be a group if and only if it does not contain any proper w. bi-ideal (i.e. a w. bi-ideal different from S).

Proof. Let S be a group and A be a w. bi-ideal of S. Let $a \in A$. Then $a(a^{-1})^2 a \in A$ i.e. e (the identity element of S) $\in A$. Let $s \in S$. Then $s = ese \in A$. So $A = S$. Hence S does not contain any proper w. bi-ideal.

Conversely, let S do not contain any proper w. bi-ideal of S. Since every one-sided ideal is a w. bi-ideal of S, it follows that S does not contain any left or right ideal. Consequently S is a group ([1] p. 6).

Proposition 1.3. If a semigroup S is left simple (right simple) then the notions of right (left) ideal and w. bi-ideal coincide.

Proof. Since in a semigroup every right ideal is a w. bi-ideal, we shall prove that in a left simple semigroup every w. bi-ideal is a right ideal. Let A be a w. bi-ideal of S. Let a_1 be an element of A. Since S is left simple $Sa = S$. Now, $a_1S = a_1Sa_1 \subseteq A$; hence A is a right ideal of S. Hence the proposition.

Proposition 1.4. Let S be a left cancellative semigroup without any idempotent. In S every w. bi-ideal is a right ideal if and only if S is left simple.

Proof. If S is left simple then from the proposition 1.3, it follows that every w. bi-ideal is a right ideal. Conversely, let in S every w. bi-ideal be a right ideal. Let $a \in S$. Then $aSa \cup \{a\}$ is a w. bi-ideal of S and hence is a right ideal of S. So, for any $x \in S$, $ax \in aSa \cup \{a\}$. If $ax = a$ then $a = ax = ax^2$ whence by left cancellation we get $x = x^2$ in S which contradicts our assumption that S is without idempotent. So $ax \in aSa$ which implies that $x \in Sa$. Hence $S = Sa$; consequently S is left simple.

We recall the following definitions.

Definition. A semigroup S is normal if Sx=xS, for all elements x of S [6].

Definition. A semigroup S is called regular if, for any element a of S there exists an element x in S such that $a = axa$ [5].

Now we have the following proposition.

Proposition 1.5. In a regular, normal semigroup every w. bi-ideal is a two-sided ideal.

Proof. Let A be a w. bi-ideal of a regular, normal semigroup S. Let $a_1 \in A$. Since S is regular $a_1 = a_1xa_1$ for some $x \in$ S. Now, $a_1\mathrm{S} = a_1xa_1S = a_1xSa_1 \subseteq A$. Similarly $Sa_1 \subseteq A$. Consequently A is a two sided ideal of S.

Proposition 1.6. If a semigroup S is left simple or right simple or is regular and normal then every w. bi-ideal is a bi-ideal of S.

Proof. The proof is similar to the proofs of the propostions 1.3 and 1.5.

§ 2. Let S be a semigroup. We denote by W (S) the set of all w. bi-ideals of S and by B (S) we mean the set of all bi-ideals of S.

Proposition 2.1. W (S) is a semigroup w.r.t. the multiplication of subsets and B (S) is an ideal of W (S).

Proof. The first part of the proposition follows from the elementary property (3) of the w. bi-ideals. From the proposition 1.1 it follows that $B(S) \subseteq W(S)$. Since the product of a bi-ideal and a nonempty subset of a semigroup S is also a bi-ideal (Theorem 8, [4]), it follows that B(S) is an ideal of W (S).

Proposition 2.2. A semigroup S is normal iff W(S) is normal.

Proof. Let S be normal. We are to show that $A.W(S)=W(S).A$, $\forall$ A $\in W(S)$. Let $X \in W(S)$ and $a \in A$. Then $aX \subseteq aS=Sa \subseteq W(S)a$. So $A.W(S) \subseteq W(S)A$. Similarly $W(S).\ A \subseteq A.W(S)$. Hence $A.W(S)=W(S).\ A$ i.e. $W(S)$ is normal. Conversely, let W(S) be normal. Let $x \in$ S. Then $xS \subseteq (x)_w\ S$ [$(x)_w=xSx \cup \{x\}$, the principal w. bi-ideal generated by x] $=A.(x)_w$ [A $\in$ W(x)] $\subseteq S(x)_w \subseteq Sx$. Similarly $Sx \subseteq xS$. Hence $xS=Sx$. So S is normal.

Proposition 2.3. A semigroup S is a semilattice of groups if and only if W(S) is a semilattice of groups.

Proof. Let S be a semigroup which is a semilattice of groups. Then S is regular and normal (Theorem 7.7 [2]). Then by the proposition 1.6 $W(S)$ and B (S) coincide. So by the theorem 7.7 of [2], it follows that $B(S)$ i. e. $W(S)$ is a semilattice of groups. Conversely, let $W(S)$ be a semilattice of groups. Then $W(S)$ is an idempotent, commutative semigroup. So $W(S)$ is regular and normal. So by the proposition 2.2, S is normal. Again, since $B(S)$ is an ideal of $W(S)$, B (S) is also regular. So S is regular (Lemma 6.4, [2]). Hence S is regular and normal so that S is a semilattice of groups.

Proposition 2.4. A semigroup S is regular if and only if $A= \bigcup\limits_{a \in A} aSa$ where A is a w. bi-ideal of S.

Proof. Let S be a regular semigroup and A be a w. bi-ideal of S. Obviously $\bigcup\limits_{a \in A} aSa \subseteq A$. Next, let $a \in A$. Since S is regular, $a=axa$ for some $x \in S$. So $a=axa \in \bigcup\limits_{a \in A} aSa$. Thus $A \subseteq \bigcup\limits_{a \in A} aSa$. Hence $A= \bigcup\limits_{a \in A} aSa$. Conversely, let $A= \bigcup\limits_{a \in A} aSa$ for any w. bi-ideal A of S. Let L be a left ideal and R be a right ideal of S. Then both L and R are w. bi-ideals of S. Since $RL \subseteq L \cap R$, $L \cap R$ is nonempty. Hence by the elementary property (2) $L \cap R$ is also a w. bi-ideal of S. Let $a \in L \cap R$. Then $a=bsb$ for some $b \in L \cap R$. Hence $a=bsb \in RL$. So $L \cap R \subseteq RL$. Thus $L \cap R=RL$; consequently S is regular.

Proposition 2.5. Let S be a semigroup. The following conditions on S are equivalent :

(1) S is regular.

(2) $A \cap B \subseteq ASB$, $A, B \in W(S)$.

(3) $(a)_w \cap (b)_w \subseteq (a)_w S (b)_w$, $(a)_w=aSa \cup \{a\}$, $a, b \in S$.

(4) $(a)_w=(a)_w S (a)_w$, $a \in S$.

Proof. (1) ⇒ (2).

Let S be regular and $A, B \in W(S)$. Let $x \in A \cap B$. Since S is regular, $x=xyx \in ASB$, for some $y \in S$. So $A \cap B \subseteq ASB$.

(2) ⇒ (3) is obvious.

(3) ⇒ (4). $(a)_w=(a)_w \cap (a)_w \subseteq (a)_w S(a)_w$. On the other hand, $(a)_w S(a)_w=(aSa \cup \{a\}) S (aSa \cup \{a\}) \subseteq aSa \subseteq (a)_w$. So $(a)_w S (a)_w=(a)_w$.

(4) ⇒ (1). Let $a \in S$. Then $a \in (a)_w=(a)_w S (a)_w$. So $a=axa$ for some $x \in S$. Thus a and hence S is regular.

Definition. A semigroup S is called completely regular if, for any element a of S, there exists an element x in S such that $a=axa$ and $ax=xa$.

Proposition 2.6. A semigroup S is completely regular if and only if every principal w. bi-ideal of S is generated by an idempotent.

Proof. Consider the w. bi-ideal $(a)_w$ of S. Since S is completely regular, $a=axa$, $ax=xa$ for some $x \in S$. Let $e=ax=xa$. Then e is an idempotent of S. We shall that $(a)_w=(e)_w$. Now $(a)_w=aSa=axaSaxa=eaSae \subseteq eSe=(e)_w$. Again, $(e)_w=eSe=axSxa \subseteq aSa=(a)_w$. Thus $(a)_w=(e)_w$. Conversely, let every principal w. bi-ideal be generated by an idempotent. Let $a \in S$. Then $(a)_w=(e)_w$ for some idempotent e of S. Now $a \in (e)_w$ implies that $a=exa$, $x \in S$. Then $ae=ea=a$. Again, $e \in (e)_w=(a)_w$, so $e=ata$ or $e=a$. If $e=a$ then $a \in aSa^2$, and if $e=ata$, then $a=ea=ata^2 \in aSa^2$. Thus $a \in aSa^2$ for every a of S. Hence S is completely regular.

Proposition 2.7. A semigroup S is completely regular if and only if for every element a of S there exists an idempotent element e in S such that $(a)_w=R(e) L(e)$ where $R(e)$ and $L(e)$ are principal right and left ideals generated by e respectively.

Proof. We shall show that $(e)_w = R(e)\ L(e)_w$. $(e)_w = eSe \subseteq R(e)\ L(e)$. On the other hand, $R(e)\ L(e) = eS.Se \subseteq eSe = (e)_w$. So $(e)_w = R(e)\ L(e)$. Now the proposition follows from the pervious proposition.

Proposition 2.8. A semigroup S is completely regular if and only if every principal w. bi-ideal of S is a monoid.

Proof. Let S be a completely regular semigroup and $(a)_w$ be the principal w. bi-ideal of S generated by a. Then $(a)_w = (e)_w = eSe$, e is an idempotent of S. Now $eSe.\,eSe \subseteq eSe$ and e is the identity element of eSe. Hence $(a)_w$ is a monoid. Conversely, let $a \in$ S. Now $(a)_w$ is a monoid with the identity element e. Now $e \in (a)_w$ implies that either $e=a$ or $e=axa$, $x \in$ S. If $e=a$, then $a \in aSa^2$. If $e=axa$, then $a=ea=axa^2 \in aSa^2$ Thus, in both the cases $a \in aSa^2$. Since a is arbitrary, it follows that S is completely regular.

Definition. A semigroup S is called a homogroup if it has a subgroup which is at the same time a two-sided ideal of S. (For an equivalent definition see [8] [7]).

Proposition 2.9. A semigroup S is a homogroup if and only if W(S) has a zero element.

Proof. Let S be a semigroup with group-ideal G. Let $B \in W$(S). Then BG is a right ideal of S and $BG \subseteq G$. Since G is a group, it follows that $BG=G$. Similarly we have $GB=G$. Thus G is a zero element of W(S). Conversely, let W(S) have a zero element Z. Then S$Z=Z$S$=Z$, since S $\in W$(S). So Z is a two-sided ideal of S. Let $z \in Z$. Then Zz is a left ideal of S and hence $Zz \in W$(S). So $Zz=Z(Zz)=Z$. Similarly we get $zZ=Z$. Thus $Zz=Z=zZ$, $\forall\, z \in Z$. Hence Z is a subgroup of S. So S is a homogroup.

Definition. A semigroup S is called left (right) regular if, for any element a of S there exists an element x in S such that $a=xa^2$ $(a=a^2x)$.

Proposition 2.10. For a normal semigroup S the following conditions are equivalent :

(1) S is regular.

(2) S is left regular.

(3) S is right regular.

(4) S is completely regular.

(5) $a \in (aS)^n$ for every element a of S and for every integer $n \geqslant 2$.

(6) $W(S)$ is idempotent.

(7) $W(S)$ is completely regular.

(8) $W(S)$ is regular.

Proof. From the theorem 6.6 of [2] it follows that (1) to (5) are equivalent. Now we assume (5). Let $a \in S$. Since (5) $\Rightarrow$ (1), S is regular. So S is regular and normal. So $B(S)=W(S)$.

Hence by the same theorem $W(S)=B(S)$ is idempotent. Obviously $(6) \Rightarrow (7) \Rightarrow (8)$. Lastly we assume (8). Since $W(S)$ is regular, $B(S)$ is also regular, since $B(S)$ is an ideal of $W(S)$. Hence S is regular (Lemma 6.4. [2]).

§ 3. **Definition 3.1.** A w. bi-ideal A of a semigroup S is called prime if
$axa \in A \Rightarrow a \in A, \ [x \in S]$.

Proposition 3.2. Let A be a prime w. bi-ideal of S. Then $S \setminus A$ is also a prime bi-ideal of S.

Proof. Let $a \in S \setminus A$. Then $axa \in S \setminus A$ for $x \in S$, for $axa \in A \Rightarrow a \in A$, a contradiction. Again, $axa \in S \setminus A$, $x \in S$ implies that $a \in S \setminus A$ for $a \in A \Rightarrow axa \in A$, a contradiction. So $S \setminus A$ is a prime w. bi-ideal of S.

Proposition 3.3. The set of all prime w. bi-ideals of a semigroup S including the empty set forms a Boolean lattice w.r.t. the set union, intersection and complementation.

Proof. The proof follows from the above proposition and the facts that union and intersection of two prime w. bi-ideals of a semigroup S are also prime w. bi-ideals on assuming that the empty set is a prime w. bi-ideal of S.

Definition 3.4. A w. bi-ideal of a semigroup S is called semiprime if
$a^3 \in A \Rightarrow a \in A, \ [a \in S]$.

Every prime w. bi-ideal of a semigroup S is also semiprime.

Proposition 3.5. A w. bi-ideal A of a semigroup S is semiprime if and only if
$a^2 \in A \Rightarrow a \in A$.

Proof. Let A be semiprime w. bi-ideal of S and $a^2 \in A$. Then $(a^3)^3=a^9=a^2.a^5.a^3 \in A \Rightarrow a^3 \in A \Rightarrow a \in A$. Conversely, let A be w. bi-ideal of S and $a^2 \in A \Rightarrow a \in A$. Let $a^3 \in A$. Then $(a^4)^2=a^3.a^2.a^3 \in A \Rightarrow (a^2)^2=a^4 \in A \Rightarrow a^2 \in A \Rightarrow a \in A$. So A is semiprime.

Proposition 3.6. A semigroup S is completely regular if and only if every w. bi-ideal of S is semiprime.

Proof. Let S be a completely regular semigroup and A be a w. bi-ideal of S. Let $a^2 \in A$. Now $a=axa$ with $xa=ax$, $x \in S$. So $a=axa=a(xa)^3=a^2 x^3 a^2 \in A$. So A is semiprime. Conversely, let in S every w. bi-ideal of S be semiprime. Hence every left ideal and every right ideal of S is semiprime. So, by the theorem 4.3 of [1] S is a union of groups. Hence S is completely regular.

References

1 Clifford, A.H. and Preston, G.B. : The Algebraic Theory of semigroups, Vol.1, Amer. Math. Soc.

2 Kuroki, N. : On normal semigroup. Czechoslovak Math. Jour. 27 (102) 1977, Praha, 43-53,

3 Lajos, S. : Characterizations of semilattice of groups, Math. Balkanica. 3(1973), 310-311.

4 Lajos, S. : Note on (m, n)-ideals II, Proc. Japan Acad. 40(1964), 631-632.

5 Luh, J. : A characterization of regular rings, Proc. Japan Acad. 39 (1964), 741-742.

6 Schwarz, S. : A theorem on normal semigroups, Czechoslovak Math. Jour. 10 (85) 1960, 197-200.

7 Thirrin, G. : Sur les homo-groupes C.R. Acad. Sci. Paris 234 (1952), 1519—1521.

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IRREDUCIBLE REPRESENTATION OF A Γ-BANACH ALGEBRA

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Abstract : The paper obtains a necessary and sufficient condition for the irreducible/ topologically irreducible left regular representation of a Γ- algebra/ Γ- Banach algebra. It also obtains Γ- equivalence/ Γ-topological equivalence between general representation and suitable reduced left regular representation of a Γ-algebra/ Γ-Banach algebra.

1.1. Introduction

We recall the definitions of Γ-algebra, Γ-normed algebra and Γ-Banach algebra of [2] which are based on the definition of Γ-ring as based on Barnes [1].

Let V and V' be two normed linear spaces over F. V is said to be a Γ-algebra over F if for $a, b, c, \in V$; $x, y \in V'$, $\alpha \in F$

(i) $axb \in V$

(ii) $(axb)yc = ax(byc)$

(iii) $\alpha\,(axb) = (\alpha a)xb = a(\alpha x)b = ax(\alpha b)$

(iv) $ax(b+c) = axb + axc$

$a(x+y)b = axb + ayb$

$(a+b)xc = axc + bxc$

This Γ-algebra is denoted by (V, V').

If V and V' are normed linear spaces over F then Γ-algebra (V, V') is called a Γ-normed algebra over F if conditions (i) ... (iv) hold and further

(v) $\| axb \| < \| a \| \; \| x \| \; \| b \|$.

Γ-normed algebra (V, V') is called a Γ-Banach algebra if V is a Banach space.

This paper deduces conditions under which a reduced left regular representation of a Γ-algebra/ Γ-Banach algebra becomes irreducible/ topologically irreducible. It also refines the equivalence theorem of A. K. Maity [3] between general representation and left regular representation.

1.2. Some definitions and elementary results

Definition 1.2.1 Let (V, V′) be a Γ-algebra and $V_1 \subsetneq V$ and $V_1' \subseteq V'$ be such that $V_1 V_1' V_1 \subseteq V_1$ and (V_1, V_1') is a Γ-algebra with respect to the same compositions as defined in (V, V′), then (V, V′) is called a Γ-subalgebra of the Γ-algebra (V, V′).

Definition 1.2.2 Let (V, V′) and (V_1, V_1') be two Γ-algebras over F. A pair of mappings $f_1 : V \to V_1$ and $f_2 : V' \to V_1'$ is said to be a homomorphism from (V, V′) to (V_1, V_1') if

(i) $f_1(a+b) = f_1(a) + f_1(b)$

(ii) $f_1(\alpha a) = \alpha f_1(a)$

(iii) $f_1(axb) = f_1(a) f_2(x) f_1(b)$

where $a, b \in V$, $x \in V'$, $\alpha \in F$.

If f_1, f_2 are both injective, then (f_1, f_2) is a monomorphism of (V, V′) into (V_1, V_1').

Result 1.2.1 Let L (V.V′), L′ (V′, V′) denote respectively the spaces of linear mappings from V to V′ and V′ to V where V, V′ are linear spaces over F.

Then L is a Γ-algebra over F denoted by (L, L') if for $A, B \in L$, $X \in L'$, $x \in V$, $\alpha, \beta \in F$

$$(\alpha A + \beta B)(x) = \alpha A(x) + \beta B(x) \qquad \ldots \quad \ldots \qquad (1.2.1.)$$

$$(AXB)(x) = A(X(B(x))) \qquad \ldots \quad \ldots \qquad (1.2.2.)$$

If V, V′ are normed linear spaces over F and B (V, V′), B' (V′, V) denote respectively the spaces of bounded linear transformations, then

$$\| AXB \| \leqslant \| A \| \; \| X \| \; \| B \| \qquad \ldots \quad \ldots \qquad (1.2.3)$$

holds and hence (B, B') becomes a Γ-normed algebra under (1.2.1) ... (1.2.3)
Also (B, B') is a Γ-Banach algebra when V is a Banach space.

Definition 1.2.3 Let (V, V′) be a Γ-algebra over F. Any homomorphism (f_1, f_2) of (V, V′) into (L, L') is called a representation of (V, V′) in (L, L') or on V.

The representation $T = (f_1, f_2)$ is faithful if the homomorphism (f_1, f_2) is a monomorphism.

If V, V′ are normed linear spaces over F, then a homomorphism of (V, V′) into (B, B') is called a normed representation of (V, V′) into (B, B') or on V.

A normed representation $T = (f_1, f_2)$ is called bounded if $\exists$ reals β, β', s. t.

$$\| T_a \| \leqslant \beta \; \| a \|, \quad \forall a \in V$$

$$\| T'_{a'} \| \leqslant \beta' \; \| a' \|, \quad \forall a' \in V'.$$

Definition 1.2.4 Let $L_i\ (V_i, V_i')$, $L'_i\ (V'_i, V_i)$, $i=1, 2$ denote respectively the set of all linear maps from V_i to V'_i and V_i' to V_i. Let (V, V') be represented on V_1 and V_2 i.e. in (L_1, L'_1) and (L_2, L'_2) under mappings

$(a \rightarrow T_a^{(1)}, a' \rightarrow T'_{a'}{}^{(1)})$ and $(a \rightarrow T_a^{(2)}, \quad a' \rightarrow T'_{a'}{}^{(2)})$, $a \in V$, $a' \in V'$. Then the two representations are said to be algebraically Γ-equivalent if there exists a 1 - 1 linear map Z from V_2 to V_1 such that

$$T'_{a'}{}^{(1)}(T_a^{(1)} Z)=Z\,(T'_{a'}{}^{(2)}\, T_a^{(2)}), \ \forall\, a \in V, \forall a' \in V'.$$

If the representations are normed, then they are said to be topologically Γ-equivalent if there exists a homeomorphism Z between V_2 and V_1 such that

$$T'_{a'}{}^{(1)}\,(T_a^{(1)}\ Z)=Z(T'_{a'}{}^{(2)}\,T_a^{(2)}), \quad \forall\, a \in V, \quad \forall\, a' \in V'.$$

Definition 1.2.5 Let $(\mathcal{B}, \mathcal{B}')$ be a Γ-subalgebra of the Γ-algebra (L, L'). A subset $M \subseteq V$ is said to be Γ-invariant with respect to $(\mathcal{B}, \mathcal{B}')$ if

$$(f'\,(f\,(M)\,)\,) \subseteq M, \ \forall f \in \mathcal{B}, \ \forall f' \in \mathcal{B}'.$$

Result 1.2.2 Let z be a fixed vector in V and let V_z denote the set of $f'\,(f\,(z)\,)$, $\forall f \in \mathcal{B}$, $\forall f' \in \mathcal{B}'$, then $\hat{V}_z$ is a Γ-invariant subspace of V with respect to $(\mathcal{B}\ \mathcal{B}')$, where $\hat{V}_z$ is the linear span of V_z.

Definition 1.2.6 A Γ-subalgebra $(\mathcal{B}, \mathcal{B}')$ of the Γ-algebra (L, L') is said to be cyclic if there exists $z \in V$ such that $\hat{V}_z = V$.

In this case z is called a cyclic vector of the Γ-algebra (V, V').

Definition 1.2.7 A Γ-subalgebra $(\mathcal{B}, \mathcal{B}')$ of the Γ-algebra (L, L') is said to be irreducible provided (0) and V are the only Γ-invariant subspaces of V with respect to $(\mathcal{B}, \mathcal{B}')$.

Result 1.2.3 Let V, V′ be normed linear spaces F and let $(\mathcal{B}, \mathcal{B}')$ be a Γ-normed subalgebra of the Γ-normed algebra (B, B′), then the closure of a Γ-invariant subspace of V is also Γ-invariant.

Definition 1.2.8 A Γ-normed subalgebra $(\mathcal{B}, \mathcal{B}')$ of the Γ-normed algebra (B, B') is topologically cyclic if $\overline{\hat{V}_z} = V$.

In this case z is called a topologically cyclic vector of (B, B').

Definition 1.2.9 A Γ-subalgebra $(\mathcal{B}, \mathcal{B}')$ of (B, B') is said to be topologically irreducible provided (o) and V are the only closed Γ-invariant subspaces of V with respect to $(\mathcal{B}, \mathcal{B}')$.

Definition 1.2.10 $(\mathcal{B}, \mathcal{B}')$ is irreducible or topologically irreducible iff every nonzero vector in $(\mathcal{B}, \mathcal{B}')$ is cyclic or topologically cyclic.

Result 1.2.4 Let $(a \to T_a^{(1)}, a' \to T'_{a'}{}^{(1)})$ be a representation of a Γ-algebra (V, V') on V_1 i.e. in (L_1, L'_1) and let M be a Γ-invariant subspace of V_1. Let $T_a^{(1)M}$, $T'_{a'}{}^{(1)M'}$ denote respectively the restrictions of $T_a^{(1)}$ on $M \subset V_1$ and $T'_{a'}{}^{(1)}$ on $M' = T_a^{(1)}(M) \subset V'_1$, then $(a \to T_a^{(1)M}, a' \to T'_{a'}{}^{(1)M'})$ is a representation of (V, V') on V_1.

Result 1.2.5 Let $(a \to T_a^{(1)}, a' \to T'_{a'}{}^{(1)})$ be a normed representation with respect to norms of V_1 and V'_1 in the sense that $\| T_a^{(1)} \| = \sup\limits_{\| x_1 \| \leqslant 1} \| T_a^{(1)}(x_1) \|$, $x_1 \in V_1$ and $\| T'_{a'}{}^{(1)} \| = \sup\limits_{\| x'_1 \| \leqslant 1} \| T'_{a'}{}^{(1)}(x'_1) \|$, $x'_1 \in V'_1$ both exist, then $(a \to T_a^{(1)M}, a' \to T'_{a'}{}^{(1)M'})$ is a normed representation of (V, V') on M. Further $\| T_a^{(1)M} \| \leqslant \| T_a^{(1)} \|$, $\| T'_{a'}{}^{(1)M'} \| \leqslant \| T'_{a'}{}^{(1)} \|$.

$(a \to T_a^{(1)M}, a' \to T'_{a'}{}^{(1)M'})$ is called the restriction of $(a \to T_a^{(1)}, a' \to T'_{a'}{}^{(1)})$.

Result 1.2.6 Let M be a Γ-invariant subspace of V_1 with respect to the restricted representation $(a \to T_a^{(1)M}, a' \to T'_{a'}{}^{(1)M'})$. Let $M' = T_a^{(1)}(M)$.

Let $(x_1)'$ and $(\tilde{x}_1)'$ denote elements of V_1/M and V_1'/M' when $x_1 \in V_1$ and $\tilde{x}_1 \in V'_1$. Let a map $a \to T_a^{(1)V_1/M}$ of (V, V') on V_1/M be defined as

$T_a^{(1)\,V_1/M}(x'_1)=(T_a^{(1)}x_1)', (T_a^{(1)}x_1) \in V_1/M$.

Similarly, let a map $a' \to T'^{(1)\,V_1'/M'}_{a'}$ of (V', V) to V'_1/M' be defined as

$T'^{(1)\,V'_1/M'}_{a'}\ \widetilde{(x_1)}' = (T'^{(1)}_{a'}\ \widetilde{x_1})'$, $\quad (T'^{(1)}_{a'}\ \widetilde{x})' \in V_1/M$, then

V_1/M remains Γ-invariant under the above pair of maps.

Result 1.2.7 $(a \to T_a^{(1)\,V_1/M}, a' \to T'^{(1)\,V'_1/M'}_{a'})$ is a representation of (V, V') on V_1/M. This representation is called the reduction of the given representation $(a \to T_a^{(1)}, \ a' \to T'^{(1)}_{a'})$ to V_1/M.

Definition 1.2.11 Let (V, V') be a Γ-Banach algebra. A subset I of V is called a left Γ-ideal of (V, V') if it is a linear subspace of V such that $VV'I \subseteq I$. It is a right Γ-ideal of (V, V') if $IV'V \subseteq I$. I is called a two sided Γ-ideal or simply a Γ-ideal if it is both a left and a right Γ-ideal.

Maximal and minimal Γ-ideals are defined in the usual way.

Definition 1.2.12 A left Γ-ideal I of (V, V') is said to be α-modular if there exists $e \in V$ such that $V - V\alpha e \subseteq I$, for some $\alpha \in V'$.

I is called modular if $V - VV'e \subseteq I$. Right α-modular/right modular Γ-ideals are defined in a similar manner. A two sided Γ-ideal I is said to be α-modular/modular if it is α-modular/modular both as a left and a right α-modular/modular Γ-ideal of (V, V').

Result 1.2.8 Every proper α-modular/modular Γ-ideal is contained in a maximal Γ-ideal of the same type.

1.3. Special representation of Γ-algebra / Γ-normed algebra / Γ-Banach algebra associated with left Γ-ideals.

In this section, we consider the left regular representation of (V, V') in the following sense. Let σ, σ' be two bilinear injective maps from $V \times V \to V'$, $V' \times V' \to V$ respectively.

Let $\sigma(a, b) = a \circ b$ and $\sigma'(a', b') = a' \overset{\sim}{\circ} b'$. Let

$$a\,u'\,b = u' \overset{\sim}{\circ} (a \circ b), \quad u' \in V'$$

$$a'\,u\,b' = u \circ (a' \overset{\sim}{\circ} b'), \quad u \in V.$$

Let (V, V') and (V', V) be two Γ-algebras over F under the above definitions of $au'b$ and $a'ub'$. Let $A : a \to A_a$ and $A' : u' \to A'_{u'}$, $A_a \in L(V, V')$ and $A'_{u'} \in L'(V', V)$ be such that

$$A_a x = a \text{ o } x, \quad x \in V \text{ and } A'_{u'} y' = u' \text{ o } y', y' \in V'.$$

Then $T = (A, A')$ is a representation of (V, V') in (L, L'). This representation is called the left regular representation of (V, V').

If (V, V') be a Γ-normed algebra over F and if it is assumed that

$\| a \text{ o } b \| \leqslant \| a \| \; \| b \|$, $\| a' \overset{\sim}{\text{o}} b' \| \leqslant \| a' \| \; \| b' \|$, then $T = (A, A')$ is a faithful bounded normed representation of (V, V') in (B, B').

Result 1.3.1 Let M be a left Γ-ideal of the Γ-Banach algebra (V, V'). Then M always remains Γ-invariant under $(A_a, A'_{u'})$, $a \in V$, $u' \in V'$. In fact,

$(a \to A_a^{M}, \; u' \to A'^{M'}_{u'})$ is a restriction of the left regular representation $(A \to A_a, \; u' \to A'_{u'})$ to M, where $M' = A_a(M)$.

Result 1.3.2 An element $z \in M$ is cyclic for the representation $(a \to A_{a'}^{M}, \; u' \to A'^{M'}_{u'})$, $M' = A_a(M)$, iff $M = VV'z$.

The above representation is irreducible iff M is a minimal Γ-ideal.

Result 1.3.3 Let $(a \to A_a, u' \to A'_{u'})$ be a normed representation of (V, V'), then $(a \to A_a^{M}, \; u' \to A'^{M'}_{u'})$ is normed and continuous.

Result 1.3.4 Let M be a left Γ-ideal of Γ-normed algebra (V, V'), then $z \in M$ is topologically cyclic for $(a \to A_a^{M}, \; u' \to A'^{M'}_{u'})$ iff M is equal to the closure of the Γ-ideal $VV'z$. The above representation is topologically irreducible iff M is a minimal closed Γ-ideal.

Result 1.3.5 Let $(a \to A_a^{V/M}, \; u' \to A'^{V'/M'}_{u})$ be the reduction of $(a \to A_a, \; u' \to A'_{u'})$ to V/M, then the reduction is a representation. For the normed representation, if M is closed in V, then the reduction is also normed.

1.4 Quasiinvertible elements of a Γ-Banach algebra

The definition of quasiinvertible elements of a Γ-algebra / Γ-Banach algebra (V, V′) is almost parallel to that of quasiregular elements of a Γ-ring [4]. The definition is as follows :

Definition 1.4.1 Let (V, V′) be a Γ-Banach algebra. An element $a \in$ V is said to be left a'-quasiinvertible, $a' \in V'$, if there exists a unique $b \in$ V such that $a+b-ba'a=0 \in$ V. a is right a'-quasiinvertible if $a+b-aa'b=0$; a is a'-quasiinvertible if it is both left and right a'-quasiinvertible. a is quasiinvertible if it is a'-quasiinvertible for all $a' \in V'$.

We now find in the following theorem the type of elements of a Γ-Banach algebra that are a -quasiinvertible.

Theorem 1.4.1 Let (V, V′) be a Γ-Banach algebra, then each $x \in$ V such that $\| x \| < 1$ is y'-quasiinvertible, $y' \in V'$, $\| y' \| < 1$ and that y'-quasiinverse y of x is given by

$$y = -\sum_{n=0}^{\infty} w_n \text{ where}$$

$w_n = (xy'xy'x \ldots (2n+1)$ elements)

Proof : Since $\| xy'x \| \leqslant \| x \| \; \| y' \| \; \| y \|$, so each of $\| w_n \|$, $n=0, 1, 2, \ldots$ is less than 1, therefore it may be shown that the partial sums of $\sum_{n=0}^{\infty} w_n$ form a Cauchy sequence in V. Hence the above series converges to an element of V. We denote this element by $\sum_{n=0}^{\infty} w_n$ and prove that $y = -\sum_{n=0}^{\infty} w_n$ is the y'-quasiinverse of x. In fact,

$$x+y-xy'y = x - \sum_{n=0}^{\infty} w_n + \sum_{n=0}^{\infty} w_{n+1} = x - x - \sum_{n=0}^{\infty} w_{n+1} + \sum_{n=0}^{\infty} w_{n+1} = 0 \in \text{V}.$$ It may similarly be shown that

$x+y-yy'x=0 \in$ V. Hence y is the y'-quasiinverse of x.

As an application of Theorem 1.4.1, we prove

Theorem 1.4.2 Let (V, V′) be a Γ-Banach algebra. Let **L** be a proper modular Γ-ideal of (V, V′) such that V—VV′ $e \subseteq$ **L**, $e \in$ V, then there is no x in **L** such that $\| e-x \| < 1$.

Proof : If possible, let $x \in$ **L** such that $\| e-x \| < 1$, then from Theorem 1.4.1 it follows that $\bar{x}=e-x$ is y'-quasiinvertible, $y' \in V'$, $\| y' \| < 1$. Let $\bar{y}$ be the $\bar{y}'$-quasiinverse of $\bar{x}$, then $\bar{x}+\bar{y}-\bar{y}y'\bar{x}=0$. Hence $e-x+\bar{y}-\bar{y}y'(e-x)=0$. So $e=x-\bar{y}y'x-(\bar{y}-\bar{y}y'e)$. Now as $x \in \mathbf{L}$ so $\bar{y}y'x \in \mathbf{L}$, Also $\bar{y}-\bar{y}y'e \in \mathbf{L}$ (**L** being modular). So $e \in \mathbf{L}$, which implies that **L** is not proper. Hence the result follows.

Corollary : If (V, V′) be a Γ-Banach algebra, then the closure of every proper α-modular/modular Γ-ideal is proper, so that maximal α-modular/modular Γ-ideals are closed.

1.5. Irreducible/topologically irreducible left regular representation

Theorem 1.5.1 (V, V′) be a Γ-algebra. Let $(a \to A_a\,,\ a' \to A'_{a'})$ be a left regular representation of (V, V′) on V, $a \in V$, $a' \in V'$. Let **L** be a left Γideal of (V, V′) and let $(a \to A_a{}^{V/\mathbf{L}},\ a' \to A'_{a'}{}^{V'/\bar{\mathbf{L}}})$, $\bar{\mathbf{L}}=A_a(\mathbf{L})$ be the corresponding reduction of the above representation on V/**L**. Then this reduced representation is irreducible iff **L** is maximal. If (V, V′) be a Γ-Banach algebra and L is closed, then the representation is topologically irreducible iff **L** is maximal closed. If **L** is modular, then irreducibility and topological irreducibility are equivalent.

Proof : Let $\mathbf{L}_1$ be any proper left Γ-ideal of (V, V′) which contains **L**. Let $\mathbf{L}_1'$ be the image of $\mathbf{L}_1$ in the quotient ring V/**L**. So

$\mathbf{L}'_1=\{(x)' \in V/\mathbf{L}$ such that $x \in \mathbf{L}_1\}$. Thus $x'=x+\mathbf{L}$, $x \in \mathbf{L}_1$. Let $\bar{\mathbf{L}}=A_a(\mathbf{L})$.

$$A'_{a'}{}^{V'/\bar{\mathbf{L}}}\,A_a{}^{V/\mathbf{L}}(x')=A'_{a'}{}^{V'/\bar{\mathbf{L}}}\,(A_a(x))'\ (\in V'/\bar{\mathbf{L}})$$

$$=A'_{a'}{}^{V'/\bar{\mathbf{L}}}\,(a \circ x)'$$

$$\neq(a'\ \overset{\sim}{\circ}\,.(a \circ x))' \in V/L$$

$$=(aa'x)' \in V/\mathbf{L},\ aa'x \in \mathbf{L}_1.$$

Hence $\mathbf{L}_1'$ is a Γ-invariant subspace of V/L under the mappings $(A_a{}^{V/L},\ A'_{a'}{}^{V'/\bar{L}})$

Again the mapping $x \to x'$ of L_1 to L_1' is a homomorphism and is 1 – 1, as L_1 is a proper Γ-ideal containing L. Since L_1 is arbitrary, so it follows that every proper Γ-ideal containing L is mapped into a Γ-invariant subspace M′ of V/L such that
$M'=\{(x)' \in V/L$ where $x \in M \subset V\}$.

It may be readily shown that M is a left Γ-ideal of (V, V′). Next let L′ be the image of L in V/L, then as L remains Γ-invariant under $(A_a^{|L}, A'^{\bar{L}}_{a'})$, $\bar{L}=A_a(L)$, so, as above, it is seen that L′ remains Γ-invariant under $(A_a^{V/L}, A'^{V'/\bar{L}}_{a'})$ and that L′ is Γ-ideal of V/L. Since the minimum subspace of V which remains Γ-invariant under left regular representation is a left Γ-ideal of V and since M′ is any subspace of V/L which remains Γinvariant under $(A_a^{V/L}, A'^{V'/\bar{L}}_{a'})$, so $L' \subseteq M'$ and hence $L \subseteq M$. Thus it is established that there is a $1-1$ map between all left Γ-ideals of (V,V′) which contain L and all Γ-invariant subspace of V/L.

Hence it follows from the definition of irreducibility of the representation of a Γ-algebra that $(A \to A_a^{V/L}, a' - A'^{V'/\bar{L}}_{a'})$ is irreducible iff L is a maximal Γ-ideal.

Next let (V, V′) be a Γ-Banach algebra and let L be closed in V. Now to establish that the above representation is topologically irreducible iff L is maximal closed, we are merely to prove that $L_1 \supseteq L$ is closed in V iff $L_1' \in V/L$ is closed in V/L. It is known that $x \to x'$ is continuous where $x \in L$ and $x' \in V/L$. Hence if L_1' is closed in V/L, then L_1 will be closed in V. Again (V, V′) being a Γ-Banach algebra, V is a Banach space. So if L is closed in V, then V/L is a Banach space. Therefore L_1' is closed in V/L when L_1 is closed in V. Hence from the definition of topological irreducibility it follows that the representation $(a \to A_a^{V/L}, a' - A'^{V'/\bar{L}}_{a'})$ is topologically irreducible iff L is a maximal closed left Γ-ideal of (V, V′).

Next let L be modular, then every left Γ-ideal $L_1 \supseteq L$ is also modular. Again by corollary to Theorem 1.4.2 it follows that closure of a proper modular left Γ-ideal is proper. So the assumption that L is maximal among closed Γ-ideals simply means that L is maximal. Hence in this case irreducibility and topological irreducibility are same.

1.6. Equivalence of general representation and left regular representation of a Γ-algebra/Γ-Banach algebra.

Theorem 1.6.1 Let $(a \to T_a, a' \to T'_{a'})$ be an irreducible representation of a Γ-algebra

(V, V′) on V. Then there exists a maximal α-modular left Γ-ideal in (V, V′) such that the representation

$(a \to T_a, \; a' \to T'_{a'})$ is algebraically Γ-equivalent to $(a \to A_a^{V/L}, a' \to A'_{a'}{}^{V'/\bar{L}})$, $\bar{L} = A_a(L)$.

Proof : $(a \to T_a, \; a' \to T'_{a'})$ being irreducible, it is automatically cyclic. Hence from [3] it is known that there exists a α-modular left Γ-ideal such that $(a \to T_a, \; a' \to T'_{a'})$ is algebraically Γ-equivalent to the representation

$(a \to A_a^{V/L}, \; a' \to A'_{a'}{}^{V'/\bar{L}})$. So $(a \to A_a^{V/L}, \; a' \to A'_{a'}{}^{V'/\bar{L}})$ is also irreducible. Hence it follows from Theorem 1.5.1 that L is maximal. The theorem now follows by the combination of the above two results.

Theorem 1.6.2 Let L be any maximal left Γ-ideal of a Γ-algebra (V, V′), then there exists a maximal α-modular left Γ-ideal of (V, V′) such that the representations

$(a \to A_a^{V/L}, \; a' \to A'_{a'}{}^{V'/\bar{L}})$ and $(a \to A_a^{V/L_1}, \; a' \to A'_{a'}{}^{V'/\bar{L}_1})$,

$\bar{L} = A_a(L)$, $\bar{L}_1 = A_a(L_1)$; are algebraically Γ-equivalent.

If further (V, V′) is a Γ-Banach algebra and is closed, then the equivalence is topological.

Proof : L being maximal, it follows from Theorem 1.5.1 that

$(a \to A_a^{V/L}, \; a' \to A'_{a'}{}^{V'/\bar{L}})$ is irreducible and hence cyclic. So practically L_1 plays the role of L in Theorem 1.6.1 and hence the proof of the theorem follows.

References

1. Barnes, W. E. : On the Γ-rings of Nabusawa, Pacific Journal Math. 18 (1966), 411-422.
2. Bhattacharya, D. K. and Maity A. K. : Regular representation of Γ-Banach algebra, accepted for publication in the proceedings of Levi seminar on Mathematics, 1989, Department of Pure Mathematics, Calcutta University.
3. Maity, A. K. : On representation of a Γ-Banach algebra, to appear in 'Ganit'— Journal of Lucknow University, Vol. 40, 1989.
4. Ravisankar T. S. and Sukla U. S. : Structure of Γ-rings, Pacific Jour. Math. Vol. 80,2,1979.
5. Rickart C. E. : General theory of Banach algebras D. Van Nostrand Company, Inc, Princeton, New Jersey, Toronto, London, New York.

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A NOTE ON GEODESICS IN AN ASSOCIATED RIEMANNIAN SPACE

B. BARUA

(Dedicated to my revered teacher Father F. Goreux, D. Sc.)

0. Introduction

Let V_n be an n-dimensional Riemannian space, of class one, whose lines of curvature are not indeterminate. The associated space $\bar{V}_n$ is defined by the relation [1]

$$\bar{g}_{ij} = e^{2\sigma} b_{ij} \tag{0.1}$$

where b_{ij} are components of the second fundamental tensor of V_n in E_{n+1} and $\sigma = \sigma(x)$ is a scalar function of the coordinates (x^i). If V_n is symmetric in the sense of Cartan, then the Christoffel symbols of the two spaces are related by the equation [2]

$$\overline{\left\{ {h \atop ij} \right\}} = \left\{ {h \atop ij} \right\} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \bar{g}^{hp} \bar{g}_{ij} \sigma_p \tag{0.2}$$

where

$$\sigma_j = \partial\sigma / \partial x^j .$$

In the present note we have deduced a necessary and sufficient condition for a geodesic in V_n to be a geodesic in $\bar{V}_n$. Some other results regarding the geodesics have also been obtained.

1. Geodesics in V_n and $\bar{V}$

Let C be a curve in V_n defined by the parametric equation

$$x^i = x^i(s), \quad i = 1, 2, \ldots, n.$$

's' being the arc-length of C in V_n. The unit tangent vector to C in V_n is given by $\left(\frac{dx^i}{ds}\right)$.

Now,

(1.1) $$\frac{dx^i}{ds} = \frac{dx^i}{d\bar{s}} \cdot \frac{d\bar{s}}{ds}$$

(1.2) and $$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{d\bar{s}^2} \cdot \left(\frac{d\bar{s}}{ds}\right)^2 + \frac{dx^i}{d\bar{s}} \cdot \frac{d^2\bar{s}}{ds^2},$$

where $\bar{s}$ is the arc-length of C in $\bar{V}_n$.

From (1.1) and (0.1) we find

(1.3) $$\left(\frac{d\bar{s}}{ds}\right)^2 = e^{2\sigma} k_n$$

where k_n is the normal curvature of V_n in E_{n+1} in the direction of C.

Differentiating both sides of (1.3) with respect to s and then substituting the value of $\frac{d^2\bar{s}}{ds^2}$ in (1.2) we find

(1.4) $$\frac{d^2x^i}{ds^2} = e^{2\sigma} k_n \left\{ \frac{d^2x^i}{d\bar{s}^2} + (\sigma_j + \tfrac{1}{2} k_j) \frac{dx^j}{d\bar{s}} \cdot \frac{dx^i}{d\bar{s}} \right\}.$$

In view of (1.4) and (0.2) we find

(1.5) $$\mu^i = e^{2\sigma} k_n \left[\bar{\mu}^i + (\tfrac{1}{2} k_j - \sigma_j) \frac{dx^j}{d\bar{s}} \frac{dx^i}{d\bar{s}} + \bar{g}^{ip} \sigma_p \right]$$

where $\mu^i = \frac{d^2x^i}{ds^2} + \left\{ {i \atop jk} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds}$, are components of the relative curvature vector of C in V_n [3] and $\bar{\mu}^i$ is a similar expression in $\bar{V}_n$, and $k_j = \frac{1}{k_n} \frac{\partial}{\partial x^j} k_n$.

If C is a geodesic in V_n then $\mu^i = 0$. It follows at once from (1.5) that C is not, in general, a geodesic in $\bar{V}_n$. If C is a geodesic in both V_n and $\bar{V}_n$, then from (1.5) we find

(1.6) $$(\tfrac{1}{2} k_j - \sigma_j) \frac{dx^j}{d\bar{s}} \frac{dx^i}{d\bar{s}} + \bar{g}^{ip} \sigma_p = 0.$$

Transvecting both sides by $\bar{g}_{ik} \dfrac{dx^k}{d\bar{s}}$ we find

(1.7) $$k_j \frac{dx^j}{d\bar{s}} = 0.$$

It follows that the vector (k_j) is orthogonal to the curve which implies that

(1.8) $$k_n\,(x^1, \ldots\ldots x^n) = \text{constant}$$

along the curve. Also from (1.6), we get

(1.9) $$\bar{g}^{ij} \sigma_j = \sigma_j \frac{dx^j}{d\bar{s}} \frac{dx^i}{d\bar{s}}.$$

On the other hand if k_n is a constant, and since (1.9) is obvious, then from (1.5) we get

$$\mu^i = e^{2\sigma} k_n \bar{\mu}^i$$

Hence if C is a geodesic in V_n, it is so in $\bar{V}_n$ also.

Theorem 1. For a geodesic C in V_n to be a geodesic in the associated space $\bar{V}_n$, it is necessary and sufficient that the normal curveture of V_n in E_{n+1} in the direction of C is a constant.

2. Curvatures

Let y^α, $\alpha = 1, \ldots, n+1$ be the rectangular Cartesian coordinates in E_{n+1}. Then [3]

(2.1) $$\frac{d^2 y^\alpha}{ds^2} = k_n \xi^\alpha + \mu^i y^\alpha_{,i}$$

where ξ^α is the unit normal vector to V_n in E_{n+1}.

If a geodesic C in V_n is a geodesic in $\bar{V}_n$ also, then $\mu^i = 0$ and by virtue of Theorem-1

$$k_1^2 = k_n^2 = \text{constant}$$

where k_1 is the first curvature [3] of C in V_n.

Now at a point on C in E_{n+1} we can find $n+1$ mutually orthogonal unit vectors forming a local coordinate system with $\left(\dfrac{dx^i}{ds}\right)$ as the unit tangent vector. If C is a geodesic then from (2.1) we find

$$\frac{d^2y^\alpha}{ds^2} = k_n\,\xi^\alpha \quad \text{or} \quad \eta_1^\alpha = \xi^\alpha \tag{2.2}$$

where η_1 is the unit first curvature vector of C in E_{n+1}.

Applying generalized Serret-Frenent's formulae [3] we get

$$\frac{\delta\eta_1^\alpha}{\delta s} = -k_n\,\frac{dy^\alpha}{ds} + k_2\,\eta_2^\alpha \tag{2.3}$$

where η_2 is the unit second curvature vector of C in E_{n+1} and k_2 is the second curvature.

Also, we have [3]

$$\frac{\delta\eta_1^\alpha}{\delta s} = b_{lj}\,g^{lm}\,y^\alpha_{,m}\,\frac{dx^j}{ds}\,. \tag{2.4}$$

Hence

$$-k_n\,\frac{dy^\alpha}{ds} + k_2\,\eta_2^\alpha = -\,b_{lj}\,g^{lm}\,y^\alpha_{,m}\,\frac{dx^j}{ds}\,. \tag{2.5}$$

Taking scalar product with η_2 and observing that the fundamental vectors at a point of C form an orthogonal ennuple, we get

$$k_2 = -g^{lm}\,b_{lj}\,\sum_\alpha \eta_2^\alpha\;y^\alpha_{,m}\,\frac{dx^j}{ds} = 0 \;:$$

Thus we find, for C, $k_1 = k_n =$ constant and $k_2 = 0$.

Theorem 2 If a geodesic C in V_n is a geodesic in the associated space $\bar{V}_n$ also, then the first curvature of C is a constant and the second curvature is zero.

Finally, if C is a geodesic in V_n only, then from (1.5) we

$$\bar{\mu}^i = (\sigma_j - \tfrac{1}{2}\,k_j)\,\frac{dx^j}{d\bar{s}}\,\frac{dx^i}{d\bar{s}} - \bar{g}^{ij}\,\sigma_j \tag{2.7}$$

Hence the first curvature of C in $\bar{V}_n$ is given by

$$\bar{k}^2 = \bar{g}_{ij}\,\bar{\mu}^i\,\bar{\mu}^j = (a - \alpha)^2 \tag{2.8}$$

where

$$a = (\sigma_j - \tfrac{1}{2}\,k_j)\,\frac{dx^j}{ds}\,. \quad \text{and} \quad \alpha = \bar{g}^{ij}\,\sigma_i\,\sigma_j.$$

References :

[1] B. Barua : Prog. Math. Vol. I (1967) 1—4.

[2] R. K. Garai : Math. Becttuk Vol 9. (1972) 333—338.

[3] L. P. Eisenhart : Riemannian Geometry, Princetion, 1949.

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SOME RESULTS ON MID-POINT SETS

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Introduction

Let A and B be two linear sets of points.

We define the mid-point set $M(A, B)$ of A and B to be the set

$M(A, B) = \left\{ \frac{x+y}{2} : x \in A,\ y \in B \right\}$, and $M^*(A, B) = \left\{ \frac{x+y}{2} : x \in A,\ y \in B,\ x \neq y \right\}$

to be the Strict Mid-point set of A and B.

It is clear that $M^*(A, B)$ is a subset of $M(A, B)$. Randolph [2] studied some properties of the Mid-point set $M(C)$ in connection with the Cantor set C in the unit interval $[0, 1]$. He proved a remarkable result that $M(C) = [0, 1]$ although C was of Lebesgue measure zero.

In this paper we have focussed our attention to some specific properties that $M(A, B)$ inherits from A & B. We also have indicated that some set-theoretic properties need not be inherited from A and B by $M(A, B)$. Our main result in the paper is that the mid-point set $M(A, B)$ of two linear sets A and B of positive measure contains an interval. Our proof is based on Lebesgue's density theorem [1] for sets of positive measure in the real line R and on two Lemmas.

In our proof the Lebesgue measure of A will be denoted by mA. We also use the result that $m(A+x) = m(x-A) = mA$ for any $x \in R$ and any measurable set $A \subset R$ [3].

Definition

Given a set $A \subset R$, for each $x \in R$, the limit inferior and the limit superior of $\frac{m^*(A \cap [x-h,\ x+h])}{2h}$ as $h \to 0+$ are denoted by $\underline{D}(A, x)$ and $\bar{D}(A, x)$ and are respectively called the lower and upper densities of A at x, where m^* stands for Lebesgue outer measure. If these two densities are equal, their common value is denoted by $D(A, x)$ and is called the density of A at x. The main theorem on density of a measurable set at a point is the Lebesgue density theorem :

If A is measurable, then the density of A exists and is equal to 1 at every point of A except possibly for a set of measure zero.

Theorem 1.

If the linear sets A and B are closed and bounded, then $M(A, B)$ is also closed and bounded.

Proof.

Since A and B are bounded, it is evident that $M(A, B)$ is bounded. We propose to show that $M(A, B)$ is closed. Let $x \in \overline{M(A, B)}$, where $\overline{M(A, B)}$ denotes the closure of $M(A, B)$. Then there exists a sequence $\{d_n\}$ in $M(A, B)$ such that, $d_n \to x$ as $n \to \infty$. Hence there exist sequences $\{x_n\}$ and $\{y_n\}$ in A and B respectively such that,

$d_n = \frac{x_n + y_n}{2}$. Since A and B are bounded and closed, $\{x_n\}$ and $\{y_n\}$ have convergent subsequences $\left\{x_{n_i}\right\}$ and $\left\{y_{n_i}\right\}$ respectively such that, $x_{n_i} \to p \in A$ and $y_{n_i} \to q \in B$.

So we may have subsequences $\left\{x_{n_{i_k}}\right\}$ of $\left\{x_{n_i}\right\}$ and $\left\{y_{n_{i_k}}\right\}$ of $\left\{y_{n_i}\right\}$ such that

$x_{n_{i_k}} \to p$ and $y_{n_{i_k}} \to q$. Also

$\frac{x_{n_{i_k}} + y_{n_{i_k}}}{2} = d_{n_{i_k}}$. Taking $k \to \infty$, we get $\frac{p+q}{2} = x$.

So, $x \in M(A, B)$. Thus $M(A, B)$ is closed.

Remark.

If A and B are closed linear sets, then $M(A, B)$ need not be closed.

For, let $A = \{-n : n \in N\}$ and $B = \left\{n + \frac{1}{2^n} : n \in N\right\}$, where N denotes the set of natural numbers. Then $0 \in \overline{M(A, B)}$, but $0 \notin M(A, B)$.

Theorem 2.

If A and B are bounded closed linear sets and if one of A and B is dense in itself, then M (A, B) is also bounded and perfect.

Proof.

By theorem 1, M (A, B) is bounded and closed. Now, it suffices to show that M (A, B) is dense in itself. Without any loss of generality let us assume that A is dense in itself. Let $d \in$ M (A, B). Then there exist points $x \in B$, $y \in B$ such that $d = \frac{x+y}{2}$. Since A is dense in itself, for $\epsilon > 0$, chosen arbitrarily the interval $(x - \epsilon, x + \epsilon)$ intersects A in at least one point, say x′, distinct from x. Let $d' = \frac{x'+y}{2} \in$ M (A, B).

Then $| d - d' | = \left| \frac{x+y}{2} - \frac{x'+y}{2} \right| = \frac{| x - x' |}{2} < \frac{\epsilon}{2}$. Since $\epsilon > 0$ is arbitrary, d is a limit point of M (A, B). The theorem, therefore follows since d is any point of M $(A$, B).

Theorem 3.

If A and B are two linear sets of second category, then M (A, B) is also of second category.

Proof.

If possible, let M (A, B) be of first category. Then so is 2M $(A, B) - p$, where p is some fixed element in A.

Let $x \in B$ and $x \neq p$. Then $\frac{x+p}{2} \in$ M (A, B). Hence $x \in$ 2M $(A, B) - p$ and this implies that $B - \{p\} \subset$ 2M $(A, B) - p$. Therefore $B - \{p\}$ is of first category and so $(B - \{p\}) \cup \{p\} = B$ is of first category which is a contradiction. Hence M (A, B) is of second category.

Remark.

If the set A is of first category or of measure zero, then the Mid-point set M (A) of A may not be a set of first category or of measure zero.

For example let us consider the Cantor set C in $[0, 1]$, which is of first category and of measure zero. But $M(C)=[0, 1]$ [2]. Then $M(C)$ is neither of first category, nor of measure zero.

Therefore the properties of being of first category or of measure zero need not be inherited by $M(A)$ from the set A.

Theorem 4.

If A and B are two linear open sets then $M(A, B)$ is also open.

Proof.

Let $d \in M(A, B)$. Then there exist $x \in A$, $y \in B$ such that $\frac{x+y}{2}=d$. Let $\epsilon > 0$ be given. Since A is open, we can find an interval I_x of length ϵ with centre at x which is wholly in A. Similarly we can find an interval I_y of length ϵ, with centre at y, which is wholly contained in B.

Let us consider the interval $I=\left(d-\frac{\epsilon}{2}, d+\frac{\epsilon}{2}\right)$. We show that $I \subset M(A, B)$. Let $\alpha \in I$. Then $d-\frac{\epsilon}{2} < \alpha < d+\frac{\epsilon}{2}$ i.e. $x+y-\epsilon < 2\alpha < x+y+\epsilon$ i.e. $x-\epsilon < 2\alpha - y < x+\epsilon$. Therefore $2\alpha - y \in I_x \subseteq A$. Hence $\frac{2\alpha - y + y}{2} = \alpha \in M(A, B)$ Thus I is contained in $M(A, B)$ and therefore $M(A, B)$ is open.

Theorem 5.

If A and B are two linear sets of positive measure, then the Mid-point set $M(A, B)$ contains an interval.

To prove this theorem we require the following Lemmas.

Lemma 1.

If $A, B, C \subset R$ are measurable then $|m(A \cap C) - m(A \cap B)| \leqslant m(B \triangle C)$, where $B \triangle C$ represents the symmetric difference of the sets B and C.

Proof.

We have $m(A \cap C) - m(A \cap B) = m(A \cap C \cap B) + m(A \cap C \cap B') - m(A \cap B \cap C) - m(A \cap B \cap C') = m(A \cap C \cap B') - m(A \cap B \cap C')$.
Then $|m(A \cap C) - m(A \cap B)| \leqslant m(A \cap C \cap B') + m(A \cap B \cap C')$
$\leqslant m(C \cap B') + m(B \cap C') = m(B \triangle C)$.

Lemma 2.

Let $A, B \subset R$ be two sets of positive Lebesgue measure. Then the function $\omega(x) = m[A \cap (2x-B)]$ is a continuous function on R.

Proof.

Since every set of positive inner measure contains a compact subset of positive measure, we may assume that A, B are compact sets having $m(A) > 0$ and $m(B) > 0$. We set $B_{2t} = 2t - B$ for $t \in R$. Then $\omega(t) = m[A \cap B_{2t}]$. Then given $\epsilon > 0$, there exists an open set $G \supset B_{2t}$ such that $m(G - B_{2t}) < \epsilon$. Let $d = \text{g.l.b.}\ \{|x-y| : x \in B_{2t},\ y \in R-G\}$. As B_{2t} is compact, so $d > 0$. Then $B_{2t+h} = B_{2t} + h \subset G$ for $|h| < d$. Therefore $m(G - B_{2t+h}) = m(G - B_{2t}) < \epsilon$.

Also $m(B_{2t+h} \triangle B_{2t}) \leqslant m(G\text{-}B_{2t+h}) + m(G\text{-}B_{2t}) < 2\epsilon$ ...... (1)

Hence by (1) and also by Lemma 1 we obtain

$|\omega(t+h) - \omega(t)| = |m(A \cap B_{2t+h}) - m(A \cap B_{2t})| \leqslant m(B_{2t+h} \triangle B_{2t}) < 2\epsilon$

for $|h| < d$, — which proves the continuity of ω.

Proof of Theorem 5.

By Lebesgue density theorem [1] we can say that for any set S of positive measure there exists a point p such that $\underset{r \to 0}{\text{Lt}} \dfrac{m[S \cap \Gamma]}{r} = 1$, where Γ is an interval of length r with p as its centre. Now there exist density points $a \in A$ and $b \in B$. Since 'a' is density point of A, we can find an interval $\triangle_r$ of length r and with centre at 'a', such that $\underset{r \to 0}{\text{Lt}} \dfrac{m(A \cap \triangle_r)}{r} = 1$, Similarly $\underset{r \to 0}{\text{Lt}} \dfrac{m(B \cap \lambda_r)}{r} = 1$, where λ_r is an interval of length r having 'b' as its centre. Let $c = \dfrac{b+a}{2}$ and $\wedge_r = 2c - \triangle_r$. We have

$$1 = \underset{r \to 0}{\text{Lt}} \frac{m(\text{B} \cap \wedge_r)}{r} = \underset{r \to 0}{\text{Lt}} \frac{m(\text{B} \cap (2c - \triangle_r))}{r}$$

$$= \underset{r \to 0}{\text{Lt}} \frac{m((2c-\text{B}) \cap \triangle_r)}{r} = \underset{r \to 0}{\text{Lt}} \frac{m(\text{B}' \cap \triangle_r)}{r}, \text{ where } \text{B}' = 2c - \text{B}.$$

Hence for a given arbitrary small $\delta (> 0)$ there is an interval $\triangle'$ about the point 'a' as centre such that $m(\text{B}' \cap \triangle') > (1-\delta)\, l$ and $m(A \cap \triangle') > (1-\delta)\, l$, l is the length of the interval $\triangle'$. This implies that the set $(\text{B}' \cap \triangle') \cap (A \cap \triangle') = (\text{B}' \cap A) \cap \triangle'$ is a set of positive measure.

Thus $m(B' \cap A) > 0$ or $m(A \cap (2c-B)) > 0$. Hence the function $\omega(x) = m[A \cap (2x-B)]$ has the property $\omega(c) > 0$. Since ω is a continuous function (by lemma 2) we can find an interval Γ_p of length p with centre as c such that $\omega(x) > 0$ for every $x \in \Gamma_p$. This show that $A \cap (2x-B) \neq \phi$ for $x \in \Gamma_p$. Therefore there are points $y \in A$ & $z \in B$ such that $y = 2x - z$ i.e. $x = \frac{y+z}{2} \in M(A, B)$ for and $x \in \Gamma_p$.

Hence the theorem.

Theorem 6.

Let A and B be two linear sets of positive measure and C be a countable dense subset of R. Then for some $d \in C$, the set $E = \left\{ x \in A : \exists\, y \in B \text{ with } \frac{x+y}{2} = d \right\}$ has positive measure.

Proof.

If possible let E be of measure zero. Then the set

$\left\{ x \in A : \exists\ y \in B \text{ such that } \frac{x+y}{2} = d \right\}$ has measure zero. Therefore the set

$F = \left\{ x \in A : \exists\ y \in B \text{ such that } \frac{x+y}{2} = d,\ d \in C \right\}$ also has measure zero.

Let $H = A - F$. Then $m(H) = m(A) > 0$. Then by theorem 5, the set $M(B, H)$ contains an interval. Hence $C \cap M(B, H) \neq \phi$. Then for some $y \in B$, $c \in C$ and $x \in H \subseteq A$ we have $\frac{x+y}{2} = c$. Thus $x \in F$, a contradiction. Hence the result.

Remark.

From the above theorem it is clear that for some d in a countable dense subset C of R, the equality $\frac{x+y}{2} = d$ holds for an uncountable (of the power of continuum) pairs $(x, y) \in A \times B$; where both the sets A and B are of positive Lebesgue measure.

Remark.

All the theorems proved in this paper hold good for $M^*(A, B)$ excepting theorem 1. If a set A is closed, $M^*(A)$ may not be closed even if A is bounded. For example $A = [0, 1]$. But $M^*(A) = (0, 1)$

References :

1. Jeffery, R. L. : "Theory of Functions of a Real Variable", Toronto (1953), p 114.

2. Randolph, J : "Some properties of sets of the Cantor type", Journal of London Mathematical Society, Vol. 16 (1941), 38 — 42.

3. __________ : Basic Real & Abstract Analysis, Academic Press, (1968), p 239.

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SEMI SYMMETRIC RECURRENT METRIC CONNECTION ON A RIEMANNIAN MANIFOLD

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0. Introduction :

Let M be a Riemannian manifold and ∇ be the Levi-Civita connection on M. Let g be the Riemann metric tensor on M. Then as we know

(1) $(\nabla_X g)(Y, Z) = 0.$

Let $\overline{\nabla}$ be a semi-symmetric affine connection on M whose torsion tensor is given by

(2) $T(X, Y) = \Pi(Y) X - \Pi(X) Y$

where Π is a differentiable 1-form on M. $\overline{\nabla}$ is called a semi-symmetric recurrent metric connection if it satisfies

(3) $(\overline{\nabla}_X g)(Y, Z) = \mu(X) g(Y, Z)$

where μ is a differentiable 1-form on M.

In this note we have obtained a necessary and sufficient condition for the conformal curvature tensor of the connection $\overline{\nabla}$ and Weyl's conformal curvature tensor of the Riemannian manifold to be the same.

1. Let $\overline{\nabla}$ be a semi-symmetric recurrent metric connection on a Riemannian manifold M satisfying (2) and (3). If ∇ denotes the L.C. connection on M, let us write

(4) $\overline{\nabla}_X Y = \nabla_X Y + H(X, Y)$

By virtue of (3), one finds

(5) $g(H(X, Y), Z) + g(H(X, Z), Y) = -\mu(X) g(Y, Z).$

Also by virtue of (2), one finds

$$H(X, Y) - H(Y, X) = \Pi(Y)X - \Pi(X)Y. \tag{6}$$

It is easy to see from (5) and (6) that

$$H(X, Y) = \alpha(Y)X - g(X, Y)A - \tfrac{1}{2}\mu(X)Y \tag{7}$$

where

$$\alpha(X) = \Pi(X) - \tfrac{1}{2}\mu(X) \tag{8}$$

and

$$g(X, A) = \alpha(X). \tag{9}$$

Hence from (4) and (7) one can write

$$\bar{\nabla}_X Y = \nabla_X Y + \alpha(Y)X - g(X, Y)A - \tfrac{1}{2}\mu(X)Y. \tag{10}$$

Let $\bar{R}(X, Y)Z$ denote the curvature tensor of the connection $\bar{\nabla}$ and $R(X, Y)Z$ denote the Riemann curvature tensor on M. Writing

$$\bar{K}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W) \text{ and } K(X, Y, Z, W) = g(R(X, Y)Z, W)$$

for the respective covariant curvature tensors [2], we get from (10)

$$\begin{aligned}\bar{K}(X, Y, Z, W) = K(X, Y, Z, W) &+ \lambda(X, Z)\,g(Y, W) - \lambda(Y, Z)\,g(X, W) \\ &+ \lambda(Y, W)\,g(X, Z) - \lambda(X, W)\,g(Y, Z) - d\mu(X, Y)\,g(Z, W)\end{aligned} \tag{1}$$

where

$$\lambda(X, Y) = (\nabla_X \alpha)(Y) - \alpha(X)\,\alpha(Y) + \tfrac{1}{2}\alpha(A)\,g(X, Y). \tag{12}$$

If $\bar{S}(X, Y)$ and $S(X, Y)$ are the Ricci tensors of the connections $\bar{\nabla}$ and ∇ respectively, and if $\bar{r}$ and r be the respective scalar curvatures, then from (11) we get

$$\bar{S}(Y, Z) = S(Y, Z) - (n-2)\,\lambda(Y, Z) - g(Y, Z)\,\mathrm{tr.}\,\lambda + d\mu(Y, Z) \tag{13}$$

and

$$\bar{r} = r - 2(n-1)\,\mathrm{tr.}\,\lambda. \tag{14}$$

From (14) and (13) we get

$$\lambda(Y, Z) = \frac{1}{n-2}\,\{S(Y, Z) - \bar{S}(Y, Z)\} - \frac{r - \bar{r}}{2(n-1)(n-2)}\,g(Y, Z) + \frac{1}{n-2}\,d\mu(Y, Z). \tag{15}$$

By virtue of (15), the equation (11) can be written as

$$\begin{aligned}\bar{K}(X, Y, Z, W) = {} & K(X, Y, Z, W) + \frac{g(Y, W)}{n-2}\{S(X, Z) - \bar{S}(X, Z)\} \\ & - \frac{g(X, W)}{n-2}\{S(Y, Z) - \bar{S}(Y, Z)\} + \frac{g(X, Z)}{n-2}\{S(Y, W) - \bar{S}(Y, W)\} \\ & - \frac{g(Y, Z)}{n-2}\{S(X, W) - \bar{S}(X, W)\} - \frac{r - \bar{r}}{(n-1)(n-2)}\{g(Y, W)\, g(X, Z) - g(X, W)\, g(Y, Z)\} \\ & + \frac{1}{n-2}\{d\mu(X, Z)\, g(Y, W) - d\mu(Y, Z)\, g(X, W) + d\mu(Y, W)\, g(X, Z) \\ & - d\mu(X, W)\, g(Y, Z) - (n-2)\, d\mu(X, Y)\, g(Z, W)\},\end{aligned}$$

which, after rearrangement, gives

$$\begin{aligned}(16)\quad \bar{C}(X, Y, Z, W) = {} & C(X, Y, Z, W) + \frac{1}{n-2}\{d\mu(X, Z)\, g(Y, W) - d\mu(Y, Z)\, g(X, W) \\ & + d\mu(Y, W)\, g(X, Z) - d\mu(X, W)\, g(Y, Z) - (n-2)\, d\mu(X, Y)\, g(Z, W)\}\end{aligned}$$

where

$$\begin{aligned}(17)\quad C(X, Y, Z, W) = {} & K(X, Y, Z, W) - \frac{1}{n-2}\{S(Y, Z)\, g(X, W) - S(X, Z)\, g(Y, W) \\ & + S(X, W)\, g(Y, Z) - S(Y, W)\, g(X, Z)\} \\ & + \frac{r}{(n-1)(n-2)}\{g(Y, Z)\, g(X, W) - g(Y, W)\, g(X, Z)\}\end{aligned}$$

is Weyl's covariant conformal curvature tensor and $\bar{C}(X, Y, Z, W)$ is a similar expression with respect to the connection $\bar{\nabla}$ representing the covariant conformal curvature tensor of $\bar{\nabla}$.

Now let μ be a closed 1-form ; then $d\mu = 0$ and it follows at once from (16) that

$$(18)\qquad \bar{C}(X, Y, Z, W) = C(X, Y, Z, W).$$

On the other hand, if the conformal curvature tensors of both the connections are equal, then from (16)

$$\begin{aligned}(19)\qquad & d\mu(X, Z)\, g(Y, W) - d\mu(Y, Z)\, g(X, W) + d\mu(Y, W)\, g(X, Z) \\ & - d\mu(X, W)\, g(Y, Z) - (n-2)\, d\mu(X, Y)\, g(Z, W) = 0.\end{aligned}$$

Contracting for Z and W, we get from (19)

$$n(n-2)\, d\mu(X, Y) = 0$$

whence

$$(20)\qquad d\mu = 0 \quad \text{for} \quad n \geqslant 3$$

that is, μ is a closed 1-form. We state the following :

Theorem : The conformal curvature tensor of a semi-symmetric recurrent metric connection on a Riemannian manifold of dimension $\geqslant 3$ is the same as Weyl's conformal curvature tensor of the manifold if and only if the differential 1-form defining the recurrence is closed.

If the 1-form μ defining the recurrence is a zero-form, then $\overline{\nabla}$ reduces to a semi-symmetric metric connection satisfying

$$(21) \qquad (\overline{\nabla}_X\, g)(Y, Z) = 0.$$

By virtue of the above theorem and (21)

$$\bar{C}(X, Y, Z, W) = C(X, Y, Z, W)$$

which is the result obtained by Imai [3].

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References :

1. Smaranda and Antonie : Ann. Fac. Sci. Uni. Nat. Zaire (Kinshasa) Sect. Math. Phys. 2 (1976), 265—270.
2. Kobayashi and Nomizu : 1963 Foundations of Differential Geometry
3. Imai T. : Tensor N. S. 24 1972, 293—296.

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THE MAXIMUM IDEMPOTENT-SEPARATING CONGRUENCE ON AN ORTHODOX Γ-SEMIGROUP

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Introduction. In [3], Sen and Saha introduced the notion of Γ-semigroup and studied some properties of regular Γ-semigroup. In a subsequent paper [4], we have defined orthodox Γ-semigroup and have investigated some of it's properties. In this paper we describe the maximum idempotent separating congruence on an orthodox Γ-semigroup.

1. Preliminaries. We recall the following definitions and results from [2], [3] and [4].

Definition. Let $M=\{a, b, c, \ldots\}$ and $\Gamma=\{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. M is called a Γ-semigroup if (i) $a \alpha b \in M$ and (ii) $(a \alpha b) \beta c = a \alpha (b \beta c)$ for all $a, b, c \in M$ and for all $\alpha, \beta, \in \Gamma$.

Definition. Let M be a Γ-semigroup. A non-empty subset B of M is said to be a Γ-subsemigroup of M if $B \Gamma B \subset B$.

Definition. Let M be a Γ-semigroup. A right (left) ideal of M is a nonempty subset I of M such that $I \Gamma M \subset I$ $(M \Gamma I \subset I)$. If I is both a right ideal and a left ideal then we say that I is an ideal of M.

Definition. Let M be a Γ-semigroup. An element $a \in$ M is said to be regular if $a \in a \Gamma \text{M} \Gamma a$, where $a \Gamma \text{M} \Gamma a = \{a \alpha b \beta a : b \in \text{M} \text{ and } \alpha, \beta \in \Gamma\}$. M is said to be regular if every element of M is regular.

Definition. Let M be a Γ-semigroup. An element $e \in$ M is said to be an idempotent in M if there exists an $\alpha \in \Gamma$ such that $e \alpha e = e$. In this case we say that e is an α-idempotent.

Definition. Let M be a Γ-semigroup and $a \in M$. Let $b \in M$ and $\alpha, \beta \in \Gamma$. b is said to be an (α, β) inverse of a if $a = a \alpha b \beta a$ and $b = b \beta a \alpha b$. In this case we shall write $b \in V_{\alpha}^{\beta}(a)$.

Analogues of Green's Relations.

Let M be a Γ-semigroup. For each element $a \in M$, the smallest right ideal containing a is called the principal right ideal generated by a and shall be denoted by $(a)_r$. We similarly define $(a)_l$ and (a), the principal left ideal and the principal ideal generated by a. We can show that

$(a)_r = a \Gamma M \cup \{a\}$, $(a)_l = M \Gamma a \cup \{a\}$ and $(a) = a \Gamma M \cup M \Gamma a \cup M \Gamma a \Gamma M \cup \{a\}$.
Let M be a Γ-semigroup. Let $a, b \in M$. Define binary relations **L, R, J, H** and **D** on M as follows :

(1) $a \mathbf{L} b$ if and only if $(a)_l = (b)_l$.
(2) $a \mathbf{R} b$ if and only if $(a)_r = (b)_r$.
(3) $a \mathbf{J} b$ if and only if $(a) = (b)$.
(4) $a \mathbf{H} b$ if and only if $a \mathbf{L} b$ and $a \mathbf{R} b$.
(5) $a \mathbf{D} b$ if and only if there exists $c \in M$ such that $a \mathbf{L} c$ and $c \mathbf{R} b$.

We note the following facts about these relations in a Γ-semigroup M.

(a) **L, R, H** and **J** are equivalence relations on M.
(b) $a \mathbf{R} b$ implies $c \alpha a \mathbf{R} c \alpha b$ for all $c \in M$ and $\alpha \in \Gamma$.
(c) $a \mathbf{L} b$ implies $a \alpha c \mathbf{L} b \alpha c$ for all $c \in M$ and $\alpha \in \Gamma$.
(d) $a \mathbf{R} b$ if and only if either $a = b$ or there exist $\alpha, \beta \in \Gamma$ and $c, d \in M$ such that $a = b \alpha c$ and $b = a \beta d$.
(e) $a \mathbf{L} b$ if and only if either $a = b$ or there exist $\alpha, \beta \in \Gamma$ and $c, d \in M$ such that $a = c \alpha b$ and $b = d \beta a$.

Lemma 1.1. Let M be a regular Γ-semigroup and let $a \in M$. Suppose e is an α-idempotent and f is a β-idempotent of M with $e \mathbf{R} a \mathbf{L} f$. Then there exists a unique $b \in V_{\beta}^{\alpha}(a)$ such that $a \beta b = e$ and $b \alpha a = f$.

Definition. A regular Γ-semigroup M is called an orthodox Γ-semigroup if e is an α-idempotent and f is a β-idempotent of M imply $e \alpha f$, $f \alpha e$ are β-idempotents of M.

Theorem 1.2. A regular Γ-semigroup M is an orthodox Γ-semigroup if and only if for $a, b \in$ M, $\alpha_1, \alpha_2, \beta_1, \beta_2, \in \Gamma$, $a' \in V^{\alpha_2}_{\alpha_1}(a)$, $b' \in V^{\beta_2}_{\beta_1}(b)$ imply that $b' \beta_2 a' \in V^{\alpha_2}_{\beta_1}(a \alpha_1 b)$ and $b' \alpha_1 a' \in V^{\alpha_2}_{\beta_1}(a \beta_2 b)$.

Lemma 1.3. Let M be an orthodox Γ-semigroup. Let $a \in$ M. If $a' \in V^{\delta}_{\gamma}(a)$, then for any α-idempotent e of M

(i) $a \gamma e \alpha a'$, $a \alpha e \gamma a'$ are δ-idempotents and

(ii) $a' \delta e \alpha a$, $a' \alpha e \delta a$ are γ-idempotents.

2. Maximum idempotent-separating congruence on an orthodox Γ-semigroup.

We first prove the following results.

Lemma 2.1. Let M be a Γ-semigroup. Let $a, b \in$ M. Then $a \mathbf{H} b$ if and only if there exist $a' \in V^{\delta}_{\gamma}(a)$, $b' \in V^{\delta}_{\gamma}(b)$ such that $a \gamma a' = b \gamma b'$, $a' \delta a = b' \delta b$.

Proof. Let $a \mathbf{H} b$. Then $a \mathbf{L} b$ and $a \mathbf{R} b$. Now $a \mathbf{L} a' \delta a$. Also $a \mathbf{R} a \gamma a'$. Again $a \mathbf{R} b$. Then $a \gamma a' \mathbf{R} b \mathbf{L} a' \delta a$. It follows from lemma 1.1. that there exists a unique $b' \in V^{\delta}_{\gamma}(b)$ such that $b \gamma b' = a \gamma a'$, $b' \delta b = a' \delta a$.

Conversely, let $a \gamma a = b \gamma b'$, $a' \delta a = b' \delta b$ for $a' \in V^{\delta}_{\gamma}(a)$, $b' \in V^{\delta}_{\gamma}(b)$. Then $a \mathbf{R} a \gamma a' = b \gamma b' \mathbf{R} b$. Hence $a \mathbf{R} b$. Again $a \mathbf{L} a' \delta a = b' \delta b \mathbf{L} b$. Then $a \mathbf{L} b$. Hence $a \mathbf{H} b$.

We have defined **L, R** and **J** in terms of principal ideals. The inclusion order among the principal ideals includes a corresponding order among the equivalence classes as follows.

$L_a \leqslant L_b$ if $\mathrm{M} \Gamma a \cup \{a\} \subset \mathrm{M} \Gamma b \cup \{b\}$.

$R_a \leqslant R_b$ if $a \Gamma \mathrm{M} \cup \{a\} \subset b \Gamma \mathrm{M} \cup \{b\}$, where M is a Γ-semigroup.

Also, we get $L_{x \gamma_1 a} \leqslant L_a$, $R_{a \gamma_2 x} \leqslant R_a$, for every $a, x \in$ M and $\gamma_1, \gamma_2 \in \Gamma$.

Definition. Let M be a Γ-semigroup. A congruence on M is defined as an equivalence relation ρ on the set M stable under left and right Γ-operation. That is for every $a, b, c \in$ M, $(a, b) \in \rho$ imply $(c \alpha a, c \alpha b) \in \rho$ and $(a \alpha c, b \alpha c) \in \rho$ for all $\alpha \in \Gamma$.

Let M be a Γ-semigroup. Let ρ be a congruence on M. We define $(a\rho)\,\alpha\,(b\rho)=(a\,\alpha\,b)\,\rho$, for all $a\rho, b\rho \in M/\rho$ and for all $\alpha \in \Gamma$. This is well defined, since for all a, a', b, b' in M and for all $\alpha \in \Gamma$, $a\rho = a'\rho$ and $b\rho = b'\rho$ imply that $(a, a') \in \rho$ and $(b, b') \in \rho$. Then $(a\,\alpha\,b,\ a'\,\alpha\,b') \in \rho$. Hence $(a\,\alpha\,b)\,\rho = (a'\,\alpha\,b')\,\rho$. It can easily by seen that M/ρ is a Γ-semigroup.

Lemma 2.2. Let M be a regular Γ-semigroup and ρ be a congruence on M. If $a\rho$ is an α-idempotent in M/ρ then there exists an α-idempotent e in M such that $a\rho = e\rho$. Moreover, e can be chosen so that $R_e \leqslant R_a$, $L_e \leqslant L_a$.

Proof. Let $a\rho$ be an α-idempotent in M/ρ. Then $a\,\rho\,\alpha\,a\,\rho = a\rho$. So, $(a\,\alpha\,a)\,\rho = a\rho$. Hence $(a, a\,\alpha\,a) \in \rho$, Let $x \in V^{\delta}_{\gamma}(a\,\alpha\,a)$. Then $(a\,\alpha\,a)\,\gamma\,x\,\delta\,(a\,\alpha\,a) = a\,\alpha\,a$ and $x\,\delta\,(a\,\alpha\,a)\,\gamma\,x = x$. Let $e = a\,\gamma\,x\,\delta\,a$. Then e is an α-idempotent. Now, $(a, a\,\alpha\,a) \in \rho$ implies that $(a\,\gamma\,x\,\delta\,a,\ a\,\gamma\,x\,\delta\,a\,\alpha\,a) \in \rho$. Again $(a, a\,\alpha\,a) \in \rho$ implies that $(a\,\gamma\,x\,\delta\,a\,\alpha\,a,\ a\,\alpha\,a\,\gamma\,x\,\delta\,a\,\alpha\,a) \in \rho$. Hence $(a\,\gamma\,x\,\delta\,a,\ a\,\alpha\,a\,\gamma\,x\,\delta\,a\,\alpha\,a) \in \rho$. Then $e\,\rho = (a\,\gamma\,x\,\delta\,a)\,\rho = (a\,\alpha\,a\,\gamma\,x\,\delta\,a\,\alpha\,a)\,\rho = (a\,\alpha\,a)\,\rho = a\,\rho$. Now, it is clear that $e = a\,\gamma\,x\,\delta\,a$ has the property $R_e \leqslant R_a$ and $L_e \leqslant L_a$.

Definition. Let M be a Γ-semigroup. A congruence ρ on M is said to be an idempotent separating congruence if e is an α-idempotent, f is an α-idempotent and $(e, f) \in \rho$ then $e = f$.

Theorem 2.3. Let M be a regular Γ-semigroup. If ρ is an idempotent separating congruence on M then $\rho \subset \mathbf{H}$.

Proof. Let ρ be an idempotent separating congruence on M. Let $(a, b) \in \rho$. Let $a' \in V^{\delta}_{\gamma}(a)$. Then $(a\,\gamma\,a',\ b\,\gamma\,a') \in \rho$ and hence $(b\,\gamma\,a')\,\rho = (a\,\gamma\,a')\,\rho$. Hence $(b\,\gamma\,a')\,\rho$ is a δ-idempotent in M/ρ since $(a\,\gamma\,a')$ is a δ-idempotent in M/ρ. By lemma 2.2 there is a δ-idempotent e in M such that $e\rho = (b\,\gamma\,a')\,\rho$ and $R_e \leqslant R_{b\,\gamma\,a'}$. Now $e\rho = (a\,\gamma\,a')\,\rho$. Hence $e = a\,\gamma\,a'$, since ρ is an idempotent separating congruence. Hence $R_a = R_{a\,\gamma\,a'} = R_e \leqslant R_{b\,\gamma\,a'} \leqslant R_b$. Similarly $R_b \leqslant R_a$. Hence $a\,\mathbf{R}\,b$. Using closely similar arguments we can show that $a\,\mathbf{L}\,b$. Hence $a\,\mathbf{H}\,b$. Hence $\rho \subset \mathbf{H}$.

Theorem 2.4. Let M be an orthodox Γ-semigroup. Define a relation μ on M by $\mu = \{ (a, b) \in M \times M :$ there exist $\gamma, \delta \in \Gamma$, $a' \in V_{\gamma}^{\delta}(a)$, $b' \in V_{\gamma}^{\delta}(b)$ satisfying $a \alpha e \gamma a' = b \alpha e \gamma b'$ and $a' \delta e \alpha a = b' \delta e \alpha b$ for every α-idempotent e of M $\}$.

Then μ is the maximum idempotent separating congruence on M.

Proof. It is immediate that μ is reflexive and symmetric. Let us prove that μ is transitive. Suppose $(a, b) \in \mu$ and $(b, c) \in \mu$. From $(a, b) \in \mu$, there exist $\gamma, \delta \in \Gamma$, $a' \in V_{\gamma}^{\delta}(a)$, $b' \in V_{\gamma}^{\delta}(b)$ such that $a \alpha e \gamma a' = b \alpha e \gamma b'$ and $a' \delta e \alpha a = b' \delta e \alpha b$ for every α-idempotent e of M. From $(b, c) \in \mu$, there exist $\gamma_1, \delta_1 \in \Gamma$, $b^* \in V_{\gamma_1}^{\delta_1}(b)$, $c^* \in V_{\gamma_1}^{\delta_1}(c)$ such that $b \beta f \gamma_1 b^* = c \beta f \gamma_1 c^*$ and $b^* \delta_1 f \beta b = c^* \delta_1 f \beta c$ for every β-idempotent f of M.

$$\begin{aligned} \text{Now, } b \gamma b' \delta a &= b \gamma b' \delta a \gamma a' \delta a \gamma a' \delta a \\ &= b \gamma b' \delta (a \gamma (a' \delta a) \gamma a') \delta a \\ &= b \gamma b' \delta b \gamma (a' \delta a) \gamma b' \delta a \qquad (a' \delta a \text{ is } \gamma\text{-idempotent}) \\ &= (b \gamma (a' \delta a) \gamma b') \delta a \\ &= a \gamma (a' \delta a) \gamma a' \delta a \\ &= a. \end{aligned}$$

Hence $b \gamma b' \delta a = a$ ... ... (1)

Similarly we can show that $b \gamma b' \delta c = c$ ... (2)

$$\begin{aligned} a \gamma_1 (b^* \delta_1 b) \gamma a' &= b \gamma_1 (b^* \delta_1 b) \gamma b' && (b^* \delta_1 b \text{ is } \gamma_1\text{-idempotent}) \\ &= b \gamma b' \quad \ldots && (\text{since } b^* \in V_{\gamma_1}^{\delta_1}(b)) \quad \ldots \ (3) \end{aligned}$$

Let $\bar{a} = b^* \delta_1 b \gamma a' \delta b \gamma b'$.

$$\begin{aligned} a \gamma_1 \bar{a} \delta a &= a \gamma_1 b^* \delta_1 b \gamma a' \delta b \gamma b' \delta a = (a \gamma_1 (b^* \delta_1 b) \gamma a') \delta (b \gamma b' \delta a) \\ &= b \gamma_1 (b^* \delta_1 b) \gamma b' \delta a && (b^* \delta_1 b \text{ is } \gamma_1\text{-idempotent and by (1)}) \\ &= b \gamma b' \delta a && (\text{since } b^* \in V_{\gamma_1}^{\delta_1}(b)) \\ &= a && (\text{by (1)}) \end{aligned}$$

$$\begin{aligned}
\bar{a}\,\delta\,a\,\gamma_1\,\bar{a} &= b^*\,\delta_1\,b\,\gamma\,a'\,\delta\,(b\,\gamma\,b'\,\delta\,a)\,\gamma_1\,b^*\,\delta_1\,b\,\gamma\,a'\,\delta\,b\,\gamma\,b' \\
&= b^*\,\delta_1\,b\,\gamma\,a'\,\delta\,(a\,\gamma_1\,(b^*\,\delta_1\,b)\,\gamma\,a')\,\delta\,b\,\gamma\,b' && (\text{by } (1)) \\
&= b^*\,\delta_1\,b\,\gamma\,a'\,\delta\,b\,\gamma\,b'\,\delta\,b\,\gamma\,b' && (\text{by } (3)) \\
&= b^*\,\delta_1\,b\,\gamma\,a'\,\delta\,b\,\gamma\,b' = \bar{a}.
\end{aligned}$$

Hence $\bar{a} \in V_{\gamma_1}^{\delta}(a)$. Next let $\bar{c} = b^*\,\delta_1\,b\,\gamma_1\,c^*\,\delta_1\,b\,\gamma\,b'$.

$$\begin{aligned}
c\,\gamma_1\,\bar{c}\,\delta\,c &= (c\,\gamma_1\,(b^*\,\delta_1\,b)\,\gamma_1\,c^*)\,\delta_1\,b\,\gamma\,b'\,\delta\,c \\
&= b\,\gamma_1\,(b^*\,\delta_1\,b)\,\gamma_1\,b^*\,\delta_1\,b\,\gamma\,b'\,\delta\,c && (b^*\,\delta_1\,b \text{ is } \gamma_1\text{-idempotent}) \\
&= b\,\gamma\,b'\,\delta\,c && (\text{since } b^* \in V_{\gamma_1}^{\delta_1}(b)) \\
&= c && (\text{by } (2))
\end{aligned}$$

Similarly $\bar{c}\,\delta\,c\,\gamma_1\,\bar{c} = \bar{c}$. Hence $\bar{c} \in V_{\gamma_1}^{\delta}(c)$.

$$\begin{aligned}
\text{Now, } a\,\alpha\,e\,\gamma_1\,\bar{a} &= (a\,\alpha\,(e\,\gamma_1\,b^*\,\delta_1\,b)\,\gamma\,a')\,\delta\,b\,\gamma\,b' \\
&= b\,\alpha\,(e\,\gamma_1\,b^*\,\delta_1\,b)\,\gamma\,b'\,\delta\,b\,\gamma\,b' && (b^*\,\delta_1\,b \text{ is } \gamma_1\text{-idempotent and } e \text{ is } \alpha\text{-idempotent, hence } e\,\gamma_1\,b^*\,\delta_1\,b \text{ is an } \alpha\text{-idempotent}) \\
&= b\,\alpha\,e\,\gamma_1\,b^*\,\delta_1\,b\,\gamma\,b' \\
&= (b\,\alpha\,(e\,\gamma_1\,b^*\,\delta_1\,b)\,\gamma_1\,b^*)\,\delta_1\,b\,\gamma\,b' \\
&= c\,\alpha\,(e\,\gamma_1\,b^*\,\delta_1\,b)\,\gamma_1\,c^*\,\delta_1\,b\,\gamma\,b' \\
&= c\,\alpha\,e\,\gamma_1\,\bar{c}.
\end{aligned}$$

$$\begin{aligned}
\bar{a}\,\delta\,e\,\alpha\,a &= b^*\,\delta_1\,b\,\gamma\,(a'\,\delta\,(b\,\gamma\,b'\,\delta\,e)\,\alpha\,a) \\
&= b^*\,\delta_1\,b\,\gamma\,b'\,\delta\,(b\,\gamma\,b'\,\delta\,e)\,\alpha\,b && (b\,\gamma\,b' \text{ is } \delta\text{-idempotent and } e \text{ is } \alpha\text{-idempotent, hence } b\,\gamma\,b'\,\delta\,e \text{ is } \alpha\text{-idempotent}) \\
&= b^*\,\delta_1\,b\,\gamma\,b'\,\delta\,e\,\alpha\,b && (\text{since } b' \in V_{\gamma}^{\delta}(b)) \\
&= b^*\,\delta_1\,b\,\gamma_1\,(b^*\,\delta_1\,(b\,\gamma\,b'\,\delta\,e)\,\alpha\,b && (\text{since } b^* \in V_{\gamma_1}^{\delta_1}(b)) \\
&= b^*\,\delta_1\,b\,\gamma_1\,c^*\,\delta_1\,(b\,\gamma\,b'\,\delta\,e)\,\alpha\,c \\
&= \bar{c}\,\delta\,e\,\alpha\,c.
\end{aligned}$$

Therefore $(a, c) \in \mu$. Hence μ is transitive.

Let $(a, b) \in \mu$ and $c \in M$. Then by definition there exist $\gamma, \delta \in \Gamma$ and $a' \in V_{\gamma}^{\delta}(a)$, $b' \in V_{\gamma}^{\delta}(b)$ such that $a \alpha e \gamma a' = b \alpha e \gamma b'$ and $a' \delta e \alpha a = b' \delta e \alpha b$ for every α-idempotent e of M. Now, $a' \in V_{\gamma}^{\delta}(a)$. Let $x \in V_{\gamma_2}^{\delta_2}(a' \delta a \beta c)$. Then it follows from theorem 1.2 that $x \delta_2 a' \in V_{\gamma_2}^{\delta}(a \gamma a' \delta a \beta c)$. Hence

$x \delta_2 a' \in V_{\gamma_2}^{\delta}(a \beta c)$ ... ... (4)

Similarly from $b' \in V_{\gamma}^{\delta}(b)$ and $x \in V_{\gamma_2}^{\delta_2}(a' \delta a \beta c)$ applying theorem 1.2 we get

$x \delta_2 b' \in V_{\gamma_2}^{\delta}(b \gamma a' \delta a \beta c)$ ... ... (5)

Now proceeding closely similar to (1) it can be shown that
$b \gamma a' \delta a = b$ ... (6) and $b' \delta a \gamma a' = b'$ ... (7) and $a' \delta a \gamma b' = b'$ ... (8)
$a \gamma a' \delta b = b$ ... (8)′

Hence from (5) $x \delta_2 b' \in V_{\gamma_2}^{\delta}(b \beta c)$ ... ... (9)

Let $g = a' \delta a \beta c \alpha e \gamma_2 x \delta_2 a' \delta a$. It follows from lemma 1.3 that $(a' \delta a \beta c) \alpha e \gamma_2 x$ is δ_2-idempotent. Also $a' \delta a$ is γ-idempotent. Hence g is γ-idempotent.

Then

$$\begin{aligned}(a \beta c) \alpha e \gamma_2 (x \delta_2 a') &= a \gamma (a' \delta a \beta c \alpha e \gamma_2 x \delta_2 a' \delta a) \gamma a' \\ &= b \gamma a' \delta a \beta c \alpha e \gamma_2 x \delta_2 a' \delta a \gamma b' \\ &= (b \beta c) \alpha e \gamma_2 (x \delta_2 b') \qquad \text{(since by (6) } b \gamma a' \delta a = b \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{and by (8) } a' \delta a \gamma b' = b')\end{aligned}$$

$$\begin{aligned}(x \delta_2 a') \delta e \alpha (a \beta c) &= x \delta_2 (a' \delta e \alpha a) \beta c = x \delta_2 b' \delta e \alpha b \beta c \\ &= (x \delta_2 b') \delta e \alpha (b \beta c).\end{aligned}$$

Thus $(a \beta c, b \beta c) \in \mu$.

Next let $y \in V_{\gamma_3}^{\delta_3}(c \beta a \gamma a')$. Also $a' \in V_{\gamma}^{\delta}(a)$. It follows from theorem 1.2 that $a' \gamma_3 y \in V_{\gamma}^{\delta_3}(c \beta a \gamma a' \delta a)$. Then $a' \gamma_3 y \in V_{\gamma}^{\delta_3}(c \beta a)$ ... (10)

Similarly from $y \in V_{\gamma_3}^{\delta_3}(c \beta a \gamma a')$ and $b' \in V_{\gamma}^{\delta}(b)$ we get $b' \gamma_3 y \in V_{\gamma}^{\delta_3}(c \beta a \gamma a' \delta b)$.

Hence $b' \gamma_3 y \in V_{\gamma}^{\delta_3}(c \beta b)$ (since by (8)′ $a \gamma a' \delta b = b$).

Then $(c\,\beta\,a)\,\alpha\,e\,\gamma\,(a'\,\gamma_3\,y) = c\,\beta\,(a\,\alpha\,e\,\gamma\,a')\,\gamma_3\,y = c\,\beta\,b\,\alpha\,e\,\gamma\,b'\,\gamma_3\,y$

$= (c\,\beta\,b)\,\alpha\,e\,\gamma\,(b'\,\gamma_3\,y)$ ... ... (11)

From lemma 1.3 it follows that $y\,\delta_3\,e\,\alpha\,(c\,\beta\,a\,\gamma\,a')$ is γ_3-idempotent. Also, $a\,\gamma\,a'$ is δ-idempotent. Hence $(a\,\gamma\,a')\,\gamma_3\,(y\,\delta_3\,e\,\alpha\,c\,\beta\,a\,\gamma\,a')$ is δ-idempotent.

$$\begin{aligned}
(a'\,\gamma_3\,y)\,\delta_3\,e\,\alpha\,(c\,\beta\,a) &= a'\,\delta\,(a\,\gamma\,a'\,\gamma_3\,y\,\delta_3\,e\,\alpha\,c\,\beta\,a\,\gamma\,a')\,\delta\,a \\
&= b'\,\delta\,(a\,\gamma\,a')\,\gamma_3\,y\,\delta_3\,e\,\alpha\,c\,\beta\,a\,\gamma\,a')\,\delta\,b \\
&= (b'\,\delta\,a\,\gamma\,a')\,\gamma_3\,y\,\delta_3\,e\,\alpha\,c\,\beta\,(a\,\gamma\,a'\,\delta\,b) \\
&= (b\,\gamma_3\,y)\,\delta_3\,e\,\alpha\,(c\,\beta\,b)
\end{aligned}$$

(since by (7) $b'\,\delta\,a\,\gamma\,a' = b'$ and by (8)$'$ $a\,\gamma\,a'\,\delta\,b = b$) ... (12)

From (11) and (12) it follows that $(c\,\beta\,a,\ c\,\beta\,b) \in \mu$. Hence μ is a congruence on M.

To show that μ is an idempotent separating congruence on M, let e and f be two α-idempotents of M such that $(e, f) \in \mu$. Then by definition there exist

$\gamma_4\ \delta_4 \in \Gamma$, $e' \in V^{\delta_4}_{\gamma_4}(e)$, $f' \in V^{\delta_4}_{\gamma_4}(f)$ such that

$e\,\nu\,g\,\gamma_4\,e' = f\,\nu\,g\,\gamma_4\,f'$ and $e'\,\delta_4\,g\,\nu\,e = f'\,\delta_4\,g\,\nu\,f$ for every ν-idempotent g of M.
Then $e\,\alpha\,e\,\gamma_4\,e' = f\,\alpha\,e\,\gamma_4\,f'$. Hence $e\,\gamma_4\,e' = f\,\alpha\,e\,\gamma_4\,f'$ ... (13)
Now, by definition $e'\,\delta_4\,e\,\alpha\,e = f'\,\delta_4\,e\,\alpha\,f$. Thus $e'\,\delta_4\,e = f'\,\delta_4\,e\,\alpha\,f$ ... (13)$'$
Hence $e'\,\delta_4\,e\,\alpha\,f = f'\,\delta_4\,e\,\alpha\,f\,\alpha\,f = f'\,\delta_4\,e\,\alpha\,f$ ... ... (14)
Again by definition $e\,\alpha\,f\,\gamma_4\,e' = f\,\alpha\,f\,\gamma_4\,f' = f\,\gamma_4\,f'$. Hence $e\,\alpha\,e\,\alpha\,f\,\gamma_4\,e' = e\,\alpha\,f\,\gamma_4\,f'$.
Thus $e\,\alpha\,f\,\gamma_4\,e' = e\,\alpha\,f\,\gamma_4\,f'$ ... ... (15)

$$\begin{aligned}
\text{Now, } e\,\alpha\,f &= e\,\gamma_4\,e'\,\delta_4\,e\,\alpha\,f && (\text{since } e' \in V^{\delta_4}_{\gamma_4}(e)) \\
&= e\,\gamma_4\,(e'\,\delta_4\,e\,\alpha\,f) = e\,\gamma_4\,(f'\,\delta_4\,e\,\alpha\,f) && (\text{by } 14) \\
&= e\,\gamma_4\,e'\,\delta_4\,e && (\text{by } 13') \\
&= e.
\end{aligned}$$

$$\begin{aligned}
e\,\alpha\,f &= e\,\alpha\,f\,\gamma_4\,f'\,\delta_4\,f && (\text{since } f' \in V^{\delta_4}_{\gamma_4}(f)) \\
&= (e\,\alpha\,f\,\gamma_4\,e')\,\delta_4\,f && (\text{by } 15) \\
&= (f\,\alpha\,f\,\gamma_4\,f')\,\delta_4\,f && (\text{by definition}) \\
&= f.
\end{aligned}$$

Hence $e = f$. Thus μ is an idempotent-separating congruence on M.

Finally, suppose that ρ is an idempotent separating congruence on M. Let $(a, b) \in \rho$. Then it follows from theorem 2.3 that $a \mathbf{H} b$. Hence from lemma 2.1, there exist

$a' \in V_{\gamma}^{\delta}(a)$, $b' \in V_{\gamma}^{\delta}(b)$ such that $a \gamma a' = b \gamma b'$, $a' \delta a = b' \delta b$.

$a' = a' \delta a \gamma a' = a' \delta b \gamma b'$ and $b' = b' \delta b \gamma b' = a' \delta a \gamma b$.

Since $(a, b) \in \rho$, $(a' \delta a, a' \delta b) \in \rho$ and accordingly $(a' \delta a \gamma b', a' \delta b \gamma b') \in \rho$.

Hence $(b', a') \in \rho$ Now $(a, b) \in \rho$. Then $(a \alpha e \gamma a', b \alpha e \gamma a') \in \rho$. Also $(a', b') \in \rho$. Then $(b \alpha e \gamma a', b \alpha e \gamma b') \in \rho$. Hence $(a \alpha e \gamma a', b \alpha e \gamma b') \in \rho$. Then $a \alpha e \gamma a' = b \alpha e \gamma b'$, since both are δ-idempotents. Similarly we can show that $a' \delta e \alpha a = b' \delta e \alpha b$ for every α-idempotent e of M. Thus $(a, b) \in \mu$. Hence $\rho \subset \mu$. Hence μ is the maximum idempotent separating congruence on M.

Note : Every semigroup can be considered as a Γ-semigroup in the following sense. Let S be an arbitrary semigroup. If S has no identity element it is easy to adjoin an extra element 1 to the set S. Then if we define $1a = a1 = a$ for all a in S and $11=1$, $S \cup \{1\}$ becomes a semigroup with identity element 1. The notion S^1 means the following.

$$S^1 = \begin{cases} S \text{ if } S \text{ has identity element} \\ S \cup \{1\} \text{ ortherwise} \end{cases}$$

S^1 is the semigroup obtained from S by adjoining an identity element if necessary. Let $\Gamma = \{1\}$. Putting $ab = a1b$ it can be shown that the semigroup S^1 is a Γ-semigroup where $\Gamma = \{1\}$. Hence we can claim that the theorem 2.4 of the paper generalizes the corresponding result of [1].

References :

1. Meakin, J.	:	Congruences on orthodox semigroup, J. Aust. Math. Soc. 12 (1971), 323—341.
2. Saha, N. K.	:	On Γ-semigroup II. Bull. Cal. Math. Soc. 79 (1987), 331—335.
3. Sen, M. K. and Saha. N. K.	:	On Γ-semigroup I. Bull. Cal. Math. Soc. 78 (1986), 180—186.
4. Sen, M. K. and Saha, N. K.	:	Orthodox Γ-semigroup. International Journal of Mathematics and Mathematical Sciences. Vol. 13 No. 3 (1990) 527—534.

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CONSTRUCTION OF ENTIRE FUNCTIONS WITH GIVEN ORDER POINTS AND TYPE POINTS REPRESENTED BY MULTIPLE DIRICHLET SERIES IN SEVERAL COMPLEX VARIABLES

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Abstract ; We consider entire functions of several complex variables represented by multiple Dirichlet series. The object of this paper is to construct some entire functions represented by multiple Dirichlet series with given order points and type points. We also discuss the convexity of a few sets associated to the order set and type set of a function in $\mathbf{C}^n$.

1. **Notations :** We denote complex and real n-space by $\mathbf{C}^n$ and $\mathbf{R}^n$ respectively and the set of non-negative integers by I, so that I^n will be the Cartesian product of n copies of I. We indicate the points $(s_1,\dots, s_n)$, $(\sigma_1,\dots,\sigma_n)$, $(m_1,\dots, m_n)$ etc. of $\mathbf{C}^n$ or $\mathbf{R}^n$ by their corresponding unsuffixed symbols s, σ, m etc. respectively.

For $s, w \in \mathbf{C}^n$ and $\alpha \in \mathbf{C}$, we define

$s = w$ iff $s_i = w_i$, $i = l, \dots, n$; $s + w = (s_l + w_l,\dots, s_n + w_n)$; $\alpha s = (\alpha s_l,\dots \alpha s_n)$: $s.w = s_l w_l + \dots + s_n w_n$; $|s| = \{|s_l|^2 + \dots + |s_n|^2\}^{1/2}$. Also for $x, y \in \mathbf{R}^n$, we say that $x \leq y$ iff $x_i, \leq y_i$, $i = l, \dots, n$; $x < y$ iff $x \leq y$ but $x \neq y$; and $x << y$ iff $x_i < y_i$, $i = l,\dots, n$.

The positive hyperoctant $\mathbf{R}^n_+$ in $\mathbf{R}^n$ will be

$$\mathbf{R}^n_+ = \{x : x \in \mathbf{R}^n,\ x_i \geqslant 0,\ i = l,\dots,n\}$$

For $s \in C^n, t \in \mathbf{R}^n_+$ we set $s^t = s_1^{t_1} \dots s_n^{t_n}$ and $\|t\| = t_1 + \dots + t_n$.

For $k \in \mathbf{R}$, $\bar{k} = (k,\dots, k)$.

An unorthodox notation : In the definition of a multiple Dirichlet series we taken sequences $\{\lambda_{jm_j}\}_{m_j=1}^{\infty}$, $(j = 1,\ldots, n)$, of exponents. We shall often require the n-tuple $(\lambda_{1m_1},\ldots, \lambda_{nm_n})$ of those sequences. For brevity we denote this tuple by $\lambda_{n,m}$ so that $s \cdot \lambda_{n,m}$ will mean $s_1 \lambda_{1m_1} + \ldots + s_n \lambda_{nm_n}$.

2. Let Ω be the family of entire functions f in $\mathbf{C}^n$ represented by a multiple Dirichlet series

$$(2.1) \qquad f(s) = \sum_{\|m\|=1}^{\infty} a_m \exp\{s \cdot \lambda_{n,m}\},$$

where $s_j = \sigma_j + i\tau_j \in \mathrm{C}\ (j = 1,\ldots, n)$, $a_m \in \mathrm{C}$ and $\{\lambda_{jm_j}\}_{m_j=1}^{\infty}$ are n sequences of exponents satisfying the conditions

$$(2.2) \qquad 0 < \lambda_{j1} < \ldots < \lambda_{jk} \to \infty \text{ as } k \to \infty, \text{ for } j = 1,\ldots, n.$$

Throughout we tacitly assume that

$$(2.3) \qquad \lim_{m_j \to \infty} \frac{\log m_j}{\lambda_{jm_j}} = 0\ (j = 1,\ldots, n).$$

we know [1] that if (2.3) holds, then the domain of convergence of (2,1) coincides with its domain of absolute convergence. Also we know [2] that the necessary and sufficient condition for the series (2.1) satisfying (2.3) to be entire is that

$$(2.4) \qquad \limsup_{\|m\| \to \infty} \frac{\log |a_m|}{\|\lambda_{n,m}\|} = -\infty$$

Corresponding to any $f \in \Omega$ we define the maximum modulus function on $\mathbf{R}^n$ by

$$M(\sigma) = M(\sigma, f) = \max_{\mathrm{re}\ s=\sigma} |f(s)|.$$

Definitions Let $f \in \Omega$ be given by (2.1). Let $B_f \subset R_+^n$ be the set of points $\alpha \in R_+^n$ such that for every $\alpha \in B_f$,

$$\text{(2.5)} \qquad \log M(\sigma) \leq e^{\sigma_1(\alpha_1+\epsilon)} + \ldots + e^{\sigma_n(\alpha_n+\epsilon)}$$

for any $\epsilon > 0$ and for sufficiently large $\|\sigma\|$, $\sigma \in R_+^n$. We call the set $\bar{B}_f$, (closure of B_f) as the order set of f. If $B_f = \phi$ (empty set), then f is said to be of infinite order. The set δB_f, the boundary of B_f, is called the order of f and any point $\rho \in \delta B_f$ is called an order point of f.

Next, let us take $\rho \in \delta B_f$, $(\rho \neq \bar{0})$ and denote by $T_f(\rho)$ the set of all $T \in R_+^n$ such that

$$\text{(2.6)} \qquad \log M(\sigma) \leq (T_1 + \epsilon)\, e^{\sigma_1 \rho_1} + \ldots + (T_n + \epsilon)\, e^{\sigma_n \rho_n}$$

for any $\epsilon > 0$ and for sufficiently large $\|\sigma\|$, $\sigma \in R_+^n$. We call the set $\overline{T_f}(\rho)$ as the type set of f corresponding to the order point ρ. If $T_f(\rho) = \phi$ (empty set), then f is said to be of infinite type. The set $\delta T_f(\rho)$ is called the type of f and any point $\tau \in \delta T_f(\rho)$ is called a type point of f corresponding to ρ.

We define the reciprocal order set of f as

$$B_f^{-1} = \{ a : a \in R_+^n,\ a_i > 0\ (i = 1, \ldots, n) \text{ such that } \left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right) \in B_f \}.$$

3. **Theorem 1.** The reciprocal order set of $f \in \Omega$ is convex.

Proof : We have shown [3] that $\alpha \in B_f\ (\alpha >> \bar{0})$, iff

$$\text{(3.1)} \qquad \limsup_{\|m\| \to \infty} \frac{\dfrac{\lambda_{1m_1}}{\alpha_1} \log \lambda_{1m_1} + \ldots + \dfrac{\lambda_{nm_n}}{\alpha_n} \log \lambda_{nm_n}}{-\log |a_m|} \leq 1$$

and $\alpha \in \delta B_t$ iff equality occurs in (3.1).

$$\text{Let } \frac{1}{\alpha}, \frac{1}{\beta} \in B_t^{-1}, \quad \text{where } \frac{1}{\alpha} = \left(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n}\right).$$

Then

$$| a_m | < \lambda_{1m_1}^{-\lambda_{1m_1} / (\alpha_1 + \epsilon)} \ldots \lambda_{nm_n}^{-\lambda_{nm_n} / (\alpha_n + \epsilon)}$$

$$\text{and } | a_m | < \lambda_{1m_1}^{-\lambda_{1m_1} / (\beta_1 + \epsilon)} \ldots \lambda_{nm_n}^{-\lambda_{nm_n} / (\beta_n + \epsilon)}$$

for all large $\| m \|$ and $\epsilon > 0$. Hence for every $t \, (0 \leq t \leq 1)$

$$| a_m | < \lambda_{1m_1}^{-\lambda_{1m_1} \left(\frac{t}{\alpha_1+\epsilon} + \frac{1-t}{\beta_1+\epsilon}\right)} \ldots \lambda_{nm_n}^{-\lambda_{nm_n} \left(\frac{t}{\alpha_n+\epsilon} + \frac{1-t}{\beta_n+\epsilon}\right)}$$

for all large $\| m \|$. This shows that

$$\frac{t}{\alpha} + \frac{1-t}{\beta} \in B_t^{-1}$$

and hence B_t^{-1} is convex.

Lemma 1. For any $\alpha \in R_+^n$ with $\alpha_i > 0$, $(i = 1, \ldots, n)$, there exists an entire multiple Dirichlet series whose order set is bounded by the hypersurface

$$\frac{\alpha_1}{\rho_1} + \ldots + \frac{\alpha_n}{\rho_n} = 1.$$

Proof : For given α, let us suppose that

$$f(s = \sum_{t=1}^{\infty} \frac{1}{t^t} \exp \{s \, . \, \lambda_{n, m}\}, \quad \lambda_{im_i} = t\,\alpha_i, \quad (i = 1, \ldots, n).$$

Then

$$a_m = t^{-t}, \text{ for those } m \text{ for which } \lambda_{im_i} = t\,\alpha_i \text{ and}$$

$$= 0, \text{ otherwise.}$$

Obviously, f is entire. Now $\rho >> \bar{0}$ is a point of the order set of f iff

$$\limsup_{\|m\| \to \infty} \frac{\frac{\lambda_{1m_1}}{\rho_1} \log \lambda_{1m_1} + \ldots + \frac{\lambda_{nm_n}}{\rho_n} \log \lambda_{nm_n}}{-\log |a_m|} \leq 1$$

that is, iff

$$\limsup_{t \to \infty} \frac{\frac{t\alpha_1}{\rho_1} \log t\alpha_1 + \ldots + \frac{t\alpha_n}{\rho_n} \log t\alpha_n}{t \log t} \leq 1$$

Hence, the order set of f is bounded by the hypersurface

$$\frac{\alpha_1}{\rho_1} + \ldots + \frac{\alpha_n}{\rho_n} = 1$$

Theorem 2. Let S be a convex set in the absolutely positive hyperoctant (i.e. every point of S is $>> \bar{0}$) such that for every point $a \in S$, all points a' with $0 < a'_i \leqslant a_i$ $(i = 1, \ldots, n)$ are in S. Then there exists an entire multiple Dirichlet series whose reciprocal order set is S.

Proof : Let the points $p_1, p_2, \ldots$ form a countable dense subset of $\delta s \in \text{int } \mathbf{R}^n{+}$.

Let us suppose that

$$(3.2) \qquad \alpha_{1i}\, x_1 + \ldots + a_{ni} x_n = 1$$

be a supporting hyperplane of S passing through p_i. By hypothesis on S, we have $\alpha_{ji} > 0$. By lemma 1, there exists an entire function

$$f_i(s) = \sum_{t=1}^{\infty} \frac{1}{t^t} \exp\{ s \,.\, \lambda_{n,\,m} \}. \; \lambda_{im_i} = t\,\alpha_i \; (i = 1, \ldots, n)$$

whose reciprocal order set is bounded by the hyperplane (3.2) and the coordinate planes. For every i we choose N_i such that

$$\bar{f}_i(s) = \sum_{t=N_i}^{\infty} \frac{1}{t^t} \exp\{ s \,.\, \lambda_{n,\,m} \} < \frac{1}{2^i}, \text{ for } R_e\, s_j < i,$$

$\lambda_{jm_j} = t\,\alpha_{ji}$ $(j = 1, \ldots, n)$, $N_i\,\alpha_{1i} + \ldots + N_i\,\alpha_{ni} > i$ and for every point $(\rho_1^{-1}, \ldots, \rho_n^{-1}) \in S$

$$| a^i_m | < \lambda_{1m_1}^{-\frac{\lambda_{1m_1}}{\rho_1+1/i}} \cdots \lambda_{nm_n}^{-\frac{\lambda_{nm_n}}{\rho_n+1/i}}$$

where a^i_m are the Dirichlet coefficients of f_i. Let us suppose that

$$f = \sum_{i=1'}^{\infty} \bar{f}_i$$

Clearly f is entire, Since all coefficients of $\bar{f}_1$ are non-negative, the maximum of $|f|$ on a poly half-plane is the sum of the maxima of $|\bar{f}_1|$ and hence the reciprocal order set of f lies in the intersection of the reciprocal order sets of $\bar{f}_i$ which is S. To prove that S is the reciprocal order set of f we write,

$$f(s) = \sum_{\| m \| = 1}^{\infty} b_m \exp\{ s \,.\, \lambda_{n,\, m} \}$$

and it suffices to show that for every $(\rho_1^{-1}, \ldots, \rho_n^{-1}) \in S$

$$\limsup_{\| m \| \to \infty} \frac{\frac{\lambda_{1m_1}}{\rho_1} \log \lambda_{1m_1} + \ldots + \frac{\lambda_{nm_n}}{\rho_n} \log \lambda_{nm_n}}{-\log | b_m |} \leq 1$$

Choose i_0 such that $\frac{1}{i_0} < \epsilon/2$ and $\| m \|$ so large that

$$\sum_{i=1}^{i_0-1} | a_m^i | < \lambda_{1m_1}^{-\frac{\lambda_{1m_1}}{\rho_1+\epsilon/2}} \cdots \lambda_{nm_n}^{-\frac{\lambda_{nm_n}}{\rho_n+\epsilon/2}}$$

Now,

$$| b_m | \leq \sum_{i=1}^{\| m \|} | a_m^i | = \sum_{i=1}^{i_0-1} | a_m^i | + \sum_{i=i_0}^{\| m \|} | a_m^i -$$

$$< \lambda_{1m_1}^{-\frac{\lambda_{1m_1}}{\rho_1+\epsilon/2}} \cdots \lambda_{nm_n}^{-\frac{\lambda_{nm_n}}{\rho_n+\epsilon/2}} + \| m \| \lambda_{1m_1}^{-\frac{\lambda_{1m_1}}{\rho_1+1/i_0}} \cdots \lambda_{nm_n}^{-\frac{\lambda_{nm_n}}{\rho_n+1/i_0}}$$

for sufficiently large $\| m \|$. So,

$$\log | b_m | < - \left(\frac{\lambda_{1m_1}}{\rho_1+\epsilon/2} \log \lambda_{1m_1} + \dots + \frac{\lambda_{nm_n}}{\rho_n+\epsilon/2} \log \lambda_{nm_n} \right) + \log (1 + \| m \|).$$

Hence,

$$\limsup_{\| m \| \to \infty} \frac{\frac{\lambda_{1m_1}}{\rho_1} \log \lambda_{1m_1} + \dots + \frac{\lambda_{nm_n}}{\rho_n} \log \lambda_{nm_n}}{- \log | b_m |} \leq 1$$

Lemma 2. Let α and ρ be two n-tuples of positive numbers satisfying

$$\frac{\alpha_1}{\rho_1} + \dots + \frac{\alpha_n}{\rho_n} = 1.$$

Also let the set of points $T \in \mathbb{R}_+^n$ be such that $T_i > 0$ $(i = 1, \dots, n)$. Then there exists an entire multiple Dirichlet series with ρ as an order point and whose type set corresponding to ρ is bounded by the hypersurface

$$\sum_{i=1}^{n} \frac{\alpha_i}{\rho_i} \log T_i = \log k \text{ (constant)}.$$

Proof. Let us consider the multiple Dirichlet series

$$f(s) = \sum_{t=1}^{\infty} \left[\frac{1}{t} \prod_{i=1}^{n} \left(\frac{\rho_i}{\alpha_i} \right)^{\alpha_i/\rho_i} ke \right]^t \exp \{ s \,.\, \lambda_{n,\,m} \}, \ \lambda_{im} = t \alpha_i.$$

Since the function f satisfies (2 4), it is entire. Also we can easily show that equality occurs in (3.1) for the function f. Hence ρ is an order point of f. We know [3] that T is a point of the type set of f with respect to the order point ρ iff

$$\limsup_{\| m \| \to \infty} \left[| a_m | \left(\frac{\lambda_{1m_1}}{e\rho_1 T_1} \right)^{\frac{\lambda_{1m_1}}{\rho_1}} \cdots \left(\frac{\lambda_{nm_n}}{e\rho_n T_n} \right)^{\frac{\lambda_{nm_n}}{\rho_n}} \right]^{\frac{1}{\| \lambda_{n \cdot m} \|}} \leq 1$$

Now, for the entire function f as constructed above, T is a point of the type set iff

$$\limsup_{t \to \infty} \left[\left\{ \frac{1}{t} \prod_{i=1}^{n} \left(\frac{\rho_i}{a_i} \right)^{\alpha_i/\rho_i} ke \right\}^t \prod_{i=1}^{n} \left(\frac{t\alpha_i}{e\rho_i T_i} \right)^{t\alpha_i/\rho_i} \right]^{\frac{1}{t \|\alpha\|}} \leq 1$$

i.e. the type set of f is bounded by the hypersurface

$$\sum_{i=1}^{n} \frac{\alpha_i}{\rho_i} \log T_i = \log k.$$

Thus the lemma is proved.

Theorem 3. Let the set $\log T = \{ (\log T_1, \ldots, \log T_n) \}$ be convex and such that all planes of support have outer normals in the positive hyper octant R^n_+. Then T is the type set of some entire multiple Dirichlet series for a prescribed order point ρ with $\rho_i > 0$, $i = 1, \ldots, n$.

Proof. Let the points $q_1, q_2, \ldots$ form a countable dense subset of points on the boundary of $\log T$. For each q_i we can find a supporting hypersurface

$$\frac{\alpha_{1i}}{\rho_1} \log T_1 + \ldots + \frac{\alpha_{ni}}{\rho_n} \log T_n = C_i, \text{ where } \alpha_{ji} > 0 \ (j = 1, \ldots, n)$$

and $C_i = 1, 0$ or -1. By lemma 2, there exists a multiple Dirichlet series

$$f_i(s) = \sum_{t=1}^{\infty} \left\{ \frac{1}{t} \prod_{j=1}^{n} \left(\frac{\rho_j}{\alpha_{ji}} \right)^{\alpha_{ji}/\rho_j} k_i e \right\}^t \exp\{ s \,.\, \lambda_{n,\,m} \},$$

$\lambda_{jm_j} = t\,\alpha_{ji}\ (j = 1, \ldots, n)$, $\log k_i = C_i$ whose type set at the point ρ is bounded by

$$\frac{\alpha_{1i}}{\rho_1} \log T_1 + \ldots + \frac{\alpha_{ni}}{\rho_n} \log T_n = \log k_i.$$

For every i we choose N_i such that

$$\bar{f}_i(s) = \sum_{t=N_i}^{\infty} \left\{ \frac{1}{t} \prod_{i=1}^{n} \left(\frac{\rho_j}{\alpha_{ji}} \right)^{\alpha_{ji}/\rho_j} k_i e \right\}^t \exp\{ s \,.\, \lambda_{n,\,m} \} < \frac{1}{2^i},$$

$\lambda_{jm_j} = t\,\alpha_{ji}\ (j = 1, \ldots, n)$ for $\operatorname{Re} s_j < i$, $N_i \alpha_{1i} + .. + N_i \alpha_{ni} > i$ and for every

$(T_1, \ldots, T_n) \notin T$,

$$|a^i_m| < \left[\frac{e\rho_1 T_1+1/i}{\lambda_{1m_1}}\right]^{\frac{\lambda_{1m_1}}{\rho_1}} \cdots \left[\frac{e\rho_n T_n+1/i}{\lambda_{nm_n}}\right]^{\frac{\lambda_{nm_n}}{\rho_n}}$$

for sufficiently large $\|m\|$ where a^i_m is the coefficient of f_i. Let $f = \sum_{i=1}^{\infty} \bar{f}_i$. Clearly f is entire, To prove the theorem it suffices to show that for every $(T_1, .. , T_n) \in T$

$$\limsup_{\|m\| \to \infty} \left[|b_m| \left(\frac{\lambda_{1m_1}}{e\rho_1 T_1}\right)^{\frac{\lambda_{1m_1}}{\rho_n}} \cdots \left(\frac{\lambda_{nm_n}}{e\rho_n T_n}\right)^{\frac{\lambda_{nm_n}}{\rho_n}}\right]^{\frac{1}{\|\lambda_n, m\|}} \leq 1,$$

where b_m's are the Dirichlet coefficients of f. We choose i_0 such that $1/i_0 < \epsilon/2$ and $\|m\|$ so large that

$$\sum_{i=1}^{i_0+1} |a^i_m| < \left[\frac{e\rho_1 T_1+\epsilon/2}{\lambda_{1m_1}}\right]^{\frac{\lambda_{1m_1}}{\rho_1}} \cdots \left[\frac{e\rho_n T_n+\epsilon/2}{\lambda_{nm_n}}\right]^{\frac{\lambda_{nm_n}}{\rho_n}}.$$

Now,

$$|b_m| \leq \sum_{i=1}^{\|m\|} |a^i_m| = \sum_{i=1}^{i_0-1} |a^i_m| + \sum_{i=i_0}^{\|m\|} |a^i_m| <$$

$$\left[\frac{e\rho_1 T_1+\epsilon/2}{\lambda_{1m_1}}\right]^{\frac{\lambda_{1m_1}}{\rho_1}} \cdots \left[\frac{e\rho_n T_n+\epsilon/2}{\lambda_{nm_n}}\right]^{\frac{\lambda_{nm_n}}{\rho_n}}$$

$$+ \|m\| \left[\frac{e\rho_1 T_1+1/i_0}{\lambda_{1m_1}}\right]^{\frac{\lambda_{1m_1}}{\rho_1}} \cdots \left[\frac{e\rho_n T_n+1/i_0}{\lambda_{nm_n}}\right]^{\frac{\lambda_{nm_n}}{\rho_n}}$$

$$\leq \left[\frac{e\rho_1 T_1+\epsilon/2}{\lambda_{1m_1}}\right]^{\frac{\lambda_{1m_1}}{\rho_1}} \cdots \left[\frac{e\rho_n T_n+\epsilon/2}{\lambda_{nm_n}}\right]^{\frac{\lambda_{nm_n}}{\rho_n}} \{1 + \|m\|\}$$

For sufficiently large $\| m \|$. Hence it follows that

$$\limsup_{\| m \| \to \infty} \left[| b_m | \left(\frac{\lambda_{1m_1}}{e\rho_1 T_1} \right)^{\frac{\lambda_{1m_1}}{\rho_1}} \cdots \left(\frac{\lambda_{nm_n}}{e\rho_n T_n} \right)^{\frac{\lambda_{nm_n}}{\rho_n}} \right]^{\frac{1}{\| \lambda_{n,m} \|}} \leq 1 .$$

This completes the proof.

References

1. Janusauskas, A. I. : Elementary theorems on the convergence of double Dirichlet series. Soviet Math. Dokl. Vol. 18, No. 3 (1977), pp 610-614.
2. Sarkar, P. K. : On Gol'dberg order and Gol'dberg type of an entire function of several complex variables represented by multiple Dirichlet series Indian J. Pure Appl. Math., 13 (10) (1982) pp 1221-1229.
3. Rahman. M. M. : On order and type of entire functions over C^2. Ganit (J. Bangladesh Math Soc.) Vol. 7, No. 1 (1987) pp 1-12.

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SYAMADAS MUKHOPADHYAYA

(1866 - 1937)

M. C. CHAKI

Syamadas Mukhopadhyaya was born on the 22nd. June, 1866 at the village Haripal in the district of Hooghly. His father Babu Gánga Kanta Mukhopadhyaya was in the judicial service. He had his early education in the village pathsala. As his father had to move from place to place in connection with his judicial duties, young Syamadas had his collegiate education at the Hooghly College (now named Hooghly Mohsin College). He took his B.A. degree from the Hooghly College and obtained his M.A. degree in mathematics from the Presidency College, Calcutta in class II in 1890. In 1909 he won the Griffiths' Prize of Rs. 900 of Calcutta University for his essay entitled " On the infinitesimal analysis of an arc ". Next year he was awarded the Ph. D. Degree in mathematics by the Calcutta University on the basis of a thesis entitled " Parametric coefficients in differential geometry of curves ". It may be mentioned that he was the first man to obatin a doctorate degree in mathematics from the Calcutta University and perhaps he was also the first Ph. D. in mathematics in India. Mukhopadhyaya joined the Bangabasi College, Calcutta as a Professor of mathematics in the same year in which he passed his M.A. examination. After serving this institution for some years he entered the Government service as a Professor of mathematics in the Bethune College, Calcutta. Here he was called upon to deliver lecture not on mathematics only but also on English literature and philosophy. In 1904 he was transferred to Presidency College, Calcutta, as Professor of mathematics and held the same post from 1904 to 1908 and again from 1909 to 1912. During the period 1908 - 1909 he was a Professor of mathematics in the Hooghly College. When the post-graduate department of Pure Mathematics was created in the Calcutta University in 1912. Sir Asutosh Mookherjee who was then the Vice-chancellor of the Calcutta University invited him to join the department and undertake the task of organising it. He accepted the invitation and did his job to the satisfaction of Sir Asutosh.

He served the department from 1912 to 1932, holding the post of a University Professor from 1926 to 1932, when he retired from the University service. After retirement he went to Europe with a Ghosh Travelling Fellowship of the Calcutta University to study methods of education in some European countries. He studied carefully their methods particularly of Fascist Italy, and on returning to India wrote a series of memoirs.

He was a founder member of the Calcutta Mathematical Society and acted as a member of the Council of the Society till 1917 when he was elected a Vice-president of the Society. After the death of Dr. Ganesh Prasad, Hardinge Professor of Higher Mathematics, Calcutta Uuiversity on the 10th. March, 1935 he was elected President of the Calcutta Mathematical Society in May, 1935 and continued as such till his death in 1937.

Prof. Mukhopadhyaya was a dedicated teacher. His affection for the students was parental. He was always ready to help them in their mathematical work. Among his students who later became noted mathematicians and / or statisticians, mention may be made of R. C. Bose, a world famous statistician who was an emeritus Professor of the State University of Colorado, U.S.A and died in 1987. In Prof. Bose's words "the kindness, encouragement and stimulus provided by Prof. Mukhopadhyaya to his students had a profound influence on their mathematical career".

Professor Mukhopadhyaya was married and had two sons and two daughters. He got the first great shock in life at the death of his eldest daughter in 1935. At the time of his death in 1937 he was survived by his wife, two sons and a daughter.

Photography was one of his hobbies. At one time he was a keen amateur photographer of no mean order. His other hobby was rose culture. He had a rose garden in his country-house at Mihijam which was unique in India because he enriched it every year by directly importing rose-saplings from England, France, Holland and other European countries.

He avoided the glamour of public life and kept himself absorbed in his own work seeking regular relaxation in his favourite rose garden at Mihijam. A man of broad sympathies and wide culture, he had an innate love for literature and philosophy. In the words of Prof. G. D. Bhar, one of his former students, "His love for the beautiful and the sublime found expression in his ardent passion for rose culture".

Contributions to Mathematics

Prof. Mukhopadhyaya was essentially a geometer. It is likely that he imbibed his love for geometry from his eminent teacher Dr. William Booth, M.A., Sc. D (Dublin) at the Hooghly College who was then well known for his geometrical researches. Mukhopadhyaya's promising potentiality in geometrical research was detected at an early age. In fact, in a very short time after taking his B.A. degree he solved an intricate geometrical problem, a fact recorded in M'clelland's 'Geometry of the Circle'.

His research contributions to geometry were outstanding for their originality and novelty of treatment. The bibliography of his published works appended at the end of this article will give an idea of the scope of his research interests.

His contributions to geometry may be broadly divided into four parts as follows :

Part (1) dealing with the properties of plane curves, specially in their infinitesimal regions by synthetic method,
Part (2) dealing with plane hyperbolic geometry,
Part (3) dealing with differential geometry of curves and
Part (4) offering a suggestion of a stereoscopic device for visualising figures in four-dimensional space.

In part (1) he developed new methods which evoked interest in certain mathematical circles abroad, for example Professors Hadamard, Engel and Blaschhke were interested in his work. These methods led to a large number of theorems on the existence of minimum number of **cyclic** and **sextactic** points between two points of a given curve, on the nature of certain ranges of points on a curve, etc. Of these theorems special mention may be made of the following two :

Theorem (i) The minimum number of **cyclic points** on a convex oval is four ;
and

Theorem (ii) the minimum number of **sextactic points** on a convex oval is six.

These two theorems published in the Bulletin of the Calcutta Mathematical Society in 1909 (Vol-I, No. 1, pp. 31—37) were rediscovered in Europe much later and have since then been the subject of investigation of a number of eminent mathematicians. Theorem (i) is now famous and is quoted in literature as **Mukhopadhyaya's four vertex theorem.** (see Differential Geometry, H. Guggenheimer, McGraw-Hill, New York, 1963 p. 30 ; Differential Geometry, J. J. Stoker, Wiley-Interscience, 1969, p. 43 ; Elements of Differential Geometry, R. S. Millman and G. D. Preker, Prentice-Hall, INC, 1977 pp. 66, 68).

The first proof of this theorem is credited to Mukhopadhyaya by W. Blaschke in his book 'Vorlesungen Uber Differential geometry, Springer, Berlin, 1924 where further references to the literature on this theorem are given.

Subsequently, Mukhopadhyaya generalized these two theorems as follows :

Theorem (i). If a circle C intersects an oval V in $2n$ points $(n \geqslant 2)$ then there exist at least $2n$ cyclic points in order on V, of alternately contrary signs, provided the oval has continuity of order 3.

Theorem (ii). If a conic C intersects an oval V in $2n$ points $(n > 2)$, then there exist at least $2n$ sextactic points, in order on V, which are alternately positive and negative, provided V has continuity of order 5.

In part (2) his contributions deal with a type of Non-Euclidean geometry, namely hyperbolic geometry. Research in Non-Euclidean geometry in India was initiated by R. Vythyanathswamy in 1914 [Journ. Ind. Math. Soc. April 1914, No. 2 pp. 58-61] Mukhopadhyaya's first paper on hyperbolic geometry appeared in 1920 and was followed by a number of other papers on this geometry. His contributions resulted in the important discovery of the rectangular pentagon and in his beautiful geometrical interpretation of the Enger-Napier rules. In collaboration with R. C. Bose and G. D. Bhar, whose names have already been mentioned, he extended to all types of hyperbolic triads of lines and points, the well known concurrency theorems of the angle bisectors and the right bisectors of the sides of an ordinary triangle.

In part (3) dealing with Differential geometry, geometry of curves he introduced for n-dimensional space curves certain differential forms called by him **parametric coefficents** by means of which he expressed many invariant properties of the curves. The distinct merit of the method of parametric coefficients of Mukhopadhyaya lies in the fact that it achieves by elementary method, results which have been obtained by advanced analysis.

Mukhopadhyaya's contribution in part (4) consist of a suggestion of a stereoscopic device for visualizing figures in four-dimensional space and his discussion with Prof. Bryan about this device. Prof. Bryan also published a paper in the Bulletin of the Calcutta Mathematical Society suggesting another device and claiming that it was superior to that of Mukhopadhyaya. A reply to this criticism was printed in the pages of the Bulletin in which Mukhopadhyaya wrote as follows :

" I do not however see any good in further prolonging the controversy between us. Both of us have fairly stated our methods. It would lie with other mathematicians interested in the problem of four dimensions to accept or reject either. "

It has already been stated that Prof. Hadamard of the Institute of France was interested in Mukhopadhyaya's work. In fact, in a letter to Makhopadhyaya, dated Feb. 23, 1923, Prof. Hadamard expressed his opinion about Mukhopadhyaya's contributions made in parts (1), (2) and (3) as follows :

" The interest of your researches on osculating conics, or even non-Euclidean geometry and parametric formulae in differential geometry of curves have been increased by comparison with memoirs published in a slightly different line by a Dane, Mr. Juel. Indeed the conjunction of both kinds of works (Mr. Juel dealing with straight lines, you with circles and conics), is likely, at my seminar, to prove of **great power** and **bearing for further improvement of geometry.** "

Prof. Mukhopadhyaya left an example of plain living and high thinking. The world of mathematics owes a great deal to him. The object of writing this article is to preserve for posterity the memoir of the inspiring life and achievements of a great Indian geometer.

BIBLIOGRAPHY

1908— A general theory of osculating conics, I. Journal, Asiatic Society, Bengal N.S. IV. 167–178

— Geometrical theory of a plane non-cyclic arc, finite as well as infinitesimal, Journal Asiatic Society, Bengal N. S. IV. 391—402.

— A general theory of osculating conics, II. Journal, Asiatic Society, Bengal N. S. IV. 497—509.

1909— New methods in the geometry of a plane arc, I Bull. Cal. Math. Soc. Vol I, 31–37.

— On rates of variation of the osculating conic, Bull. Cal. Math. Soc. Vol I 125–130.

— Parametric coefficients in the Differential geometry of curves in an N-space, I, General conceptions, Bull. Cal. Math. Soc. I, 187—200.

— Parametic cofficients in the Differential geometry of curves in an N-space, II Extension of Serret-Frenet formulae in curves in an N-dimensional space, Bull. Cal. Math. Soc. Vol. I.

1910— Parametric cofficients in the Differential geometry of curves in an N-space, III, Fundamental formulae, Bull. Cal. Math. Soc. Vol. II, 27–35.

— On the infinitesimal analysis of arc. Griffith's Prize essay, 1909. Calcutta Univ. Publication.

1911— Parametric coefficients in the Differential geometry of curves in an N-space, IV, Expressions of the coordinates of a point on a curve in an N-space as power series in X. Bull. Cal. Math. Soc. Vol. III.

1912— A note on the stereoscopic representation of four-dimensional space, Bull. Cal. Math. Soc. Vol. IV 15.

1913— Parametric coefficients in the Differential geometry of curves in an N-space. V. Principal directions and curvatures at a singular points of a curve in an N-space, Bull. Cal. Math. Soc. Vol. V, 13–20.

1914— Reply to Professor Bryan's criticism, Bull. Cal. Math. Soc. Vol. VI, 55—56.

1915— Parametric coefficients in the Differential geometry of curves in an N-space. VI. On null-parametric coefficients and osculating spheres to a curve in an N-space, Bull. Cal. Math. Soc. Vol. VII.

1917— A note on current view of operations through the fourth dimension, Vol. IX, 19—21.

1919— New methods in the geometry of a plane arc-II, Cyclic points and normals. Bull. Cal. Math. Soc. Vol. X, 65—72.

1920— (With G. D. Bhar) Generalization of certain theorems in the hyperbolic geometry of the triangle, Bull. Cal. Math. Soc. Vol. XII, 15—28.

1922— Geometrical investigations on the correspondence between a right-angled, a three-right-angled quadrilateral and a rectangular pentagon in hyperbolic geometry, Bull Cal. Math. Soc. Vol. XIII, 211—216.

1922— Some general theorems in the geometry of a plane curve, Sir Asutosh Mookherjee, Silver Jubilee Vol. II. Calcutta University Publication.

1926— Genesis of an elementary arc, Bull. Cal. Math. Soc. Vol. XVII, 153—158.

— General theorems of co-intimacy of symmetrics of a hyperbolic triad, Bull. Cal. Math. Soc. Vol XVII, 39—48.

1927— (With R. C. Bose). Triadic equations in hyperbolic geometry, Bull. Cal. Math. Soc. Vol. XVIII, 99—110.

— Note on T. Hayashi's paper on the osculating ellipses of a plane curve. Rend. Ciroolo Matematico di palermo, Tomo LI.

1929— Generalized form of Bohmer's theorem for elliptically curled non-analytic oval, Math. Zeit. Band 30, 560—571.

1931— Extended minimum number theorems of cyclic and sextactic points on a plane convex oval, Math. Zeit. Band 33, 648—662.

— Circles incident on an oval of undefined curvature, Tohoku Math. Journ. Vol. 37.

1932— Lower segments of M-curves, Journ. Ind. Math. Soc. Vol. 19, 75—80.

— Cyclic curves on an ellipsoid, Journ. Ind. Math. Soc. Vol. 19, 246—250.

— An exposition of the axioms of order of Hilbert, Indian Physico-Math Journ. Vol. III.

1933— Gemattic extensions of elementary chains, Tohoku Math. Journ. Vol. 38

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Applications of operational calculus : definite integrals and summations of series

N. K. CHAKRAVARTY

1. In the present note we list certain definite integrals and series summations involving special functions derived from the application of certain rules and sequences obtained for the operational calculus based on the multidimensional Laplace transformations. Since we are interested in their applications to definite integrals and series summations, we simply state these rules and sequences omitting their derivations which are easy consequences of known results of different dimensional operational calculus.

The n-dimensional Laplace transformation of F is defined as usual by

$f(\mathbf{p})=\mathbf{p}\int_0^\infty \exp(-\mathbf{p},\mathbf{x})\,F(\mathbf{x})\,d\mathbf{x}$, where $f(\mathbf{p})=f(p_1, p_2, \ldots, p_n)$ with a similar definition for $F(\mathbf{x})$, $\mathbf{p}=p_1\,p_2 \cdot p_n$, $(\mathbf{p}.\mathbf{x})=\sum_{i=1}^{n} p_i x_i$, and $d\mathbf{x}=dx_1\,dx_2 \ldots dx_n$ the integral being over an n-dimensional infinite Block, The convergence problem of the n-dimensional Laplace transformation occurs in the author's paper [1]. We call $F(\mathbf{x})$ the original, $f(\mathbf{p})$ the image and the operational relation between F and f is indicated by the notations.

$f \underset{n}{\subset} F$ or $F \underset{n}{\supset} f$; for n=2 by $f \subset\subset F$ or $F \supset\supset f$; for n=1 by $f \subset F$ or $F \supset f$. For operational relations in one and more variables and the special functions used in the context we refer mostly to the standard works of Erdelyi et al [2], Voelker and Doetsch [3], Ditkin and Prudnikov [4] and Poli and Delerue [5]. Our scheme of presentation is as follows. We state the relevant rules/sequences first and derive the desired integrals by briefly indicating their derivations from these. The conditions of validity of the integrals are stated. Derivations of summation formulae are briefly stated.

2. Definite integrals involving Bessel's functions

If $f(\mathbf{p}) \underset{n}{\subset} h(\mathbf{x})$, then $x^{\nu-n} f(\mathbf{x}) \supset \Gamma(\nu+1)\, p \int_0^\infty h(\mathbf{u})/(p+u_1+\cdots+u_n)^{\nu+1}\, d\mathbf{u}$, re ν, re $p>0$, the integral being over the infinite Block.

Definite integrals

(i) From the known operational representations for $J_o\left((x^2+2xy)^{\frac{1}{2}}\right)$ and $x^{\nu}/(1+x^2)$ (See [2], [3]), we obtain

$$\int_0^\infty\int_0^\infty J_o\left((2u^2+2uv)^{\frac{1}{2}}\right)(u+v+a)^{\nu+1}\, du\, dv = \pi \csc(\nu+1)\pi)/\Gamma(\nu+1)\ V_{\nu+1}(2a,0)$$

where $V_\nu(w,z)$ is the Lommel function of two variables, re $p>0$, re $a>0$, re$\nu>-\frac{1}{4}$.

(ii) Let $h(x,y)=J_o\left(2(x+y)^{\frac{1}{2}}\right)$, so that $f(p,q)=(pe^{-1/q}-qe^{-1/p})/(p-q)$.

Then $f(x,x)=\lim\limits_{y\to x}(y\,e^{-1/x}-x\,e^{-1/y})/(y-x)=e^{-1/x}-e^{-1/x}/x,$

Therefore

$$\int_0^\infty\int_0^\infty J_o\left(2(u+v)^{\frac{1}{2}}\right)/(u+v+a)^{\nu+1}\, du\, dv$$

$$= 2/\Gamma(\nu+1)\ a^{-\frac{1}{2}(\nu-1)}\left(K_{\nu-1}\ (2a^{\frac{1}{2}}) - a^{\frac{1}{2}})\, K_{\nu-2}\ 2a^{\frac{1}{2}})\right),\ \text{re } a>0,\ \text{re } \nu > \tfrac{1}{4}$$

(iii) Let $h(x,y,z) = J_o\left(2(x+y+z)^{\frac{1}{2}}\right)$, then.

$$f(p, q, r) = qr / (q-p)(r-p)e^{-1/p} + rp / (r-q)(p-q)\ e^{-1/q} + pq / (p-r)(q-r)e^{-1/r}.$$

Putting $r=p+h$, $q=p+k$ and then making $h, k \to o$, it follows that

$f(p, p, p) = e^{-1/p}(1-1/p)$; and therefore

$$\int_o^\infty\int_o^\infty\int_o^\infty J_o\,(2(u+v+w)^{\frac{1}{2}}) / (u+v+w+a)^{\nu+1}\,du\,dv\,dw$$

$$= 2 / \Gamma\ (\nu+1)\ a^{-\frac{1}{2}(\nu-2)}\,(K_{\nu-2}\,(2a^{\frac{1}{2}}) - a^{\frac{1}{2}} K_{\nu-3}\,(2a^{\frac{1}{2}})),\ \text{re } a>o,\ \text{re } \nu > -\tfrac{1}{4}.$$

(iv) Let $h\ (x,y) = \pi^{-\frac{1}{2}}(x+y)^{-\frac{1}{2}}e^{-xy / (x+y)}$, so that

$f\,(p,q) = (pq)^{\frac{1}{2}}(p^{\frac{1}{2}} + q^{\frac{1}{2}})/((p^{\frac{1}{2}}+q^{\frac{1}{2}})^2 + 1)$. Then

$$\int_o^\infty\int_o^\infty \exp\,(--uv / (u+v))\,(u+v)^{-\frac{1}{2}}(u+v+a)^{-\nu-1}\,du\,dv$$

$$=\pi^{\frac{1}{2}}\ \Gamma(\nu+\tfrac{1}{2}) / 4^{\nu}\,\Gamma\,(\nu+1)\,e^{a/4}\ \Gamma\,(\tfrac{1}{2}-\nu,\ a / 4),\ \text{re } a >o,\ \text{re } \nu>-\tfrac{1}{2}\ ;\ \Gamma(-\nu, p)$$

denotes that incomplete Γ-function.

Rule 2. If $f(p) \underset{n}{\subset} h(x)$ and

$F(x) = \int_o^\infty J_\lambda^\mu((a, x)\ x^{\mu)})h(x)\,dx$, where $J_\lambda^\mu\,(.)$ denotes the generalized Bessel function,

then $p^{n\nu-\lambda} f\,(a/p^\mu) \subset ax^\lambda\ F(x)$, $\mu>o$, re $\lambda > -1$, a's are positive constants and re $p>0$.

When $\mu=1$, we have the following version of the rule.

If $f(p) \underset{n}{\subset} h\,(x)$ and $F(x) = \int_o^\infty J_\lambda\,(2(a, x)^{\frac{1}{2}}\ x^{\frac{1}{2}}) / (a, x)^{\frac{1}{2}}\ h(x)\,dx$, then

$p^{n-\lambda}\ f(a / p) \subset a\ x^{\frac{1}{2}\lambda}\ F(x)$, λ, p and a satisfy conditions as before.

Definite integrals.

(i) Let $f(p, q) = p^{1-\nu} q / pq+1)^{1+\nu}$, so that $h(x,y) = x^{\nu} / \Gamma(\nu+1) J_{2\nu} (2(xy)^{\frac{1}{2}})$

re $\nu > -\frac{1}{2}$. Then

$$\int_0^\infty \int_0^\infty J_2 \left(2(au+bv)^{\frac{1}{2}} x^{\frac{1}{2}}\right) J_0 \left(2(uv)^{\frac{1}{2}}\right) / (au+bv)\, du\, dv = (abx)^{-1} \left(1 - \cos\left((ab)^{\frac{1}{2}} x\right)\right),\ a,\ b > 0$$

and

$$\int_0^\infty \int_0^\infty J_{3\nu} \left(2(au+bv)^{\frac{1}{2}} x^{\frac{1}{2}}\right) u^{\nu} J_{2\nu} \left(2(uv^{\frac{1}{2}})\right) / (au+bv)^{-3/2\nu} du\ dv$$

$$= \pi^{\frac{1}{2}}\, 2^{-\nu-\frac{1}{2}}\, a^{-\frac{1}{2}(3\nu-\frac{1}{2})}\, b^{-\frac{1}{2}(\nu-\frac{1}{2})}\, x^{-\frac{1}{2}(\nu-1)}\, J_{\nu-\frac{1}{2}} \left((ab)^{\frac{1}{2}} x\right),\ a,b > 0,\ \text{re}\ \nu > 1.$$

(ii) Let $f(p,q) = pq\,(p+q)\left((1+p^2)\ (1+q^2)\right)^{-\frac{1}{2}} \left((1+p^2)^{\frac{1}{2}} + (1+q^2)^{\frac{1}{2}}\right)^{-1}$

so that $h(x,y) = J_0 (x+y)$.

Then $$\int_0^\infty \int_0^\infty J_1 \left(2(u+v)^{\frac{1}{2}} x^{\frac{1}{2}}\right) J_0\ (u+v) / (u+v)^{\frac{1}{2}} du\ dv = x^{\frac{1}{2}} J_1 (x)$$

and $$\int_0^\infty \int_0^\infty J_2 \left(2(u+v)^{\frac{1}{2}} x^{\frac{1}{2}}\right) J_0\ (u+v) / (u+v)\ du\ dv$$

$= \frac{1}{2} \pi (J_1 (x) H_0 (x) - J_0 (x) H_1 (x))$, H_0, Struve's function.

(iii) Let $f(p,q) = pq\,(p+q)\left((1+p^2)\ (1+q^2)\right)^{-\frac{1}{2}} \left(p(1+q^2)^{\frac{1}{2}} + q(1+p^2)^{\frac{1}{2}}\right)^{-1}$;

so that $h(x,y) = J_1 (x+y)$. Then

$$\int_0^\infty \int_0^\infty J_1 \left(2(u+v)^{\frac{1}{2}} x^{\frac{1}{2}}\right) J_1(u+v) / (u+v)^{\frac{1}{2}} du\ dv = x^{\frac{1}{2}} J_0\ (x);$$

$$\int_0^\infty \int_0^\infty J_2 (2(u+v)^{\frac{1}{2}} x^{\frac{1}{2}}) J_1 (u+v) / (u+v) \, du \, dv = J_1 (x) ;$$

$$\int_0^\infty \int_0^\infty J_3 (2(u+v)^{\frac{1}{2}} x^{\frac{1}{2}}) J_1 (u+v) / (u+v)^{3/2} \, du \, dv = \pi / 2x^{\frac{1}{2}} (J_1 (x) H_0 (x) - J_0 (x) H_1(x))$$

(iv) Let $f(p,q) = pq / (p^2 + 1 + (q-p)(p^2 + 1)^{\frac{1}{2}}$; so that

$h(x,y) = Jo\ ((x^2 + 2xy^{\frac{1}{2}}$. Then

$$\int_0^\infty \int_0^\infty J_2 (2(u+v)^{\frac{1}{2}} x^{\frac{1}{2}}) J_0 ((u^2+2uv)^{\frac{1}{2}}) / (u+v) \, du dv = (1-\cos x) / x.$$

3. Definite integrals involving confluent hypergeometric functions.

Rule. If $f(p) \subset h(x)$, $h(x^2) / x \supset \phi(p)$, re $p>0$,

then $\phi(p) = 2p / \pi \int_0^\infty \int_0^\infty \exp (-\frac{1}{4} p^2 s^2)) f (t^2+s^{-2}) / (1+s^2t^2) \, ds \, dt.$

Definite integrals

(i) Taking $h(x) = e^{-x} x^{n+\frac{1}{2}}$, we have

$$\int_0^\infty \int_0^\infty \exp (-a^2 s^2) s^{2n+1} / (1+s^2+s^2t^2) n+3 / 2 \, ds \, dt$$

$$= \pi^{\frac{1}{2}} 2^{n-\frac{1}{2}} \Gamma(n+1) / (2n+1) e^{\frac{1}{2}a^2} D_{-2n-1} (a2^{\frac{1}{2}})$$

$a>0$, $n>-\frac{1}{2}$,

(ii) Let $h(x)=x^{\frac{1}{2}} / (1+x)^{\nu+\frac{1}{2}}$ and $k=-\frac{1}{2}\nu$, where $-5 / 4 < \text{re}\ k < -1 / 4$.

Then $\int_0^\infty \int_0^\infty \exp (-a^2 s^2+\frac{1}{2} (t^2+s^{-2})) s^{2k+1} w_{k,k+1} (t^2+a^{-2}) / (1+s^2t^2)^{k+8}$

$=\pi / 2a^{2k+1} \Gamma(2k+3 / 2) (H_{2k+1} (2a) - Y_{2k+1} (2a))$, $a>0$.

Rule 2. If $\phi(p,q) \subset\subset f(x,y)$ and $\Psi(p) \subset g(x)$, then

$$\int_0^\infty \phi(u,v)\ g(u)\ /\ uv\ du = \int_0^\infty\int_0^\infty f(x,y)\ \Psi\ (x) \exp\ (-vy)\ /\ \mathbf{x}\ dx\ dy$$

Definite integrals

Let $f(x,y) = x^{\lambda-\mu} y^{\mu}\ /\ \Gamma(\lambda+1)\ J_{2\mu}(\ 2(xy)^{\frac{1}{2}}),\ g(x) = \exp\ (-1/x)\ x^{-1-2\nu}$

$$\text{Then} \int_0^\infty\int_0^\infty x^{\lambda-\mu+\nu}\ y^{\mu} \cdot J_{2\mu}\ (2(xy)^{\frac{1}{2}})\ k_{2\nu}\ (2x^{\frac{1}{2}})\ e^{-vy}\ dx\ dy$$

$$= \tfrac{1}{2}\Gamma(\lambda+1)\ \Gamma(\lambda+2\nu+1)\ v^{\lambda+\nu-2\mu-\frac{1}{2}} e^{\frac{1}{2}v}\ W_{-\lambda-\nu-\frac{1}{2},\nu}\ (v),\ \text{re}\ \mu,\ \text{re}\ (\lambda-\mu),\ \text{re}\ (\lambda-\mu+2\nu) > -\tfrac{1}{2}$$

$$\text{(ii)} \int_0^\infty\int_0^\infty x^{-\mu-\frac{1}{2}}\ y^{\mu}\ J_{2\mu}\ (2(xy)^{\frac{1}{2}})\ K_{2\nu}\ (2x^{\frac{1}{2}})\ e^{-vy}\ dx\ dy$$

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}}\Gamma\ (\tfrac{1}{2}+\nu)\ \Gamma(\tfrac{1}{2}-\nu)\ v^{-2\mu-\frac{1}{2}}\ e^{\frac{1}{2}v}\ K_{\nu}(\tfrac{1}{2}v).$$

4. Convolution formulae

It is well-known (see(3)) that if $f_j\ (x,y) \supset\supset \phi_j\ (p,q)$, $j=1,2$, then under suitable conditions

$$1/pq\ \phi_1\ (p,q)\ \phi_2\ (p,q) \subset\subset \int_0^x\int_0^y f_1\ (x-\lambda,\ y-\rho)\ f\ (\lambda,\rho)\ d\lambda\ d\rho$$

$$= \int_0^x\int_0^y f_1\ (\lambda,\rho)\ f_2\ (x-\lambda)\ y-\rho)\ d\lambda\ d\rho = f_1\ (x,y) * * f_2\ (x,y),$$

the convolution formula for the two dimensional operational calculus.

The ster and stei functions are defined by means of the relation

$H_0\ (z\ i^{3/2}) = \text{ster}\ z + i\ \text{stei}\ z$, where H_0 is the Struve function having the operational representation $H_0 (2\ (xy)^{\frac{1}{2}}) \supset\supset\ (pq)^{\frac{1}{2}}/(1+pq)$.

Hence $\begin{matrix}\text{ster}\\ \text{stei}\end{matrix}\ (2(xy)^{\frac{1}{2}}) \supset\supset 2^{-\frac{1}{2}}\ (pq)^{\frac{1}{2}}\ (1 \pm pq)\ /\ (1+p^2\ q^2)$.

By using the well-known operational representations for $I_0(2(xy)^{\frac{1}{2}})$, $J_0(2(xy)^{\frac{1}{2}})$, $\begin{smallmatrix}\mathrm{ber}\\ \mathrm{bei}\end{smallmatrix}(2(xy)^{\frac{1}{2}})$, we have the following convolution formulae.

(i) $(\mathrm{ster}\,(2(xy)^{\frac{1}{2}}) - \mathrm{stei}\,(2(xy)^{\frac{1}{2}})) ** 1 = \mathrm{ster}\,(2(xy)^{\frac{1}{2}}) + \mathrm{stei}\,(2(xy)^{\frac{1}{2}})$;

(ii) $(xy)^{\frac{1}{2}} ** \mathrm{ber}\,(2(xy)^{\frac{1}{2}}) = \pi / 4\cdot 2^{\frac{1}{2}}\,(\mathrm{ster}\,(2(xy)^{\frac{1}{2}}) + \mathrm{stei}\,(2(xy)^{\frac{1}{2}}))$;

(iii) $8.2^{\frac{1}{2}} / 9\pi\,((xy)^{3/2} ** (\mathrm{ster}\,(2(xy)^{\frac{1}{2}}) + \mathrm{stei}\,(2(xy)^{\frac{1}{2}}))$

$= 32.2^{\frac{1}{2}} / 225\pi\,((xy)^{5/2} ** (\mathrm{ster}\,(2(xy)^{\frac{1}{2}}) - \mathrm{stei}\,(2(xy)^{\frac{1}{2}}))$

$= \frac{1}{4}x^2y^2 + \mathrm{ber}(2(xy)^{\frac{1}{2}}) - 1.$

(iv) $2.2^{\frac{1}{2}} /\pi\,(\,(xy)^{\frac{1}{2}} ** (\mathrm{ster}\,(2(xy)^{\frac{1}{2}}) + \mathrm{stei}\,(2\,(xy)^{\frac{1}{2}})\,)$

$= 8.2^{\frac{1}{2}}/9\pi\,(\,(xy)^{3/2} ** (\mathrm{ster}\,(2(xy)^{\frac{1}{2}}) - \mathrm{stei}\,(2(xy)^{\frac{1}{2}})\,)$

$= xy - \mathrm{bei}\,(2(xy)^{\frac{1}{2}}).$

(v) $(xy)^{3/2} ** \begin{smallmatrix}\mathrm{ster}\\ \mathrm{stei}\end{smallmatrix}(2(xy)^{\frac{1}{2}})$

$= 9\pi.2^{\frac{1}{2}}/32\,(\frac{1}{4}x^2y^2 \pm xy + \mathrm{ber}\,(2(xy)^{\frac{1}{2}}) \mp \mathrm{bei}\,(2(xy)^{\frac{1}{2}}) - 1).$

(vi) $1/2\,(\mathrm{ster}\,(2(xy)^{\frac{1}{2}}) + \mathrm{stei}\,(2(xy)^{\frac{1}{2}})\,)$

$= J_0(2(xy)^{\frac{1}{2}}) ** \mathrm{ster}\,(2(xy)^{\frac{1}{2}})$

$= -I_0(2(xy)^{\frac{1}{2}}) ** \mathrm{stei}\,(2(xy)^{\frac{1}{2}}).$

5. **Self-reciprocal functions.** A special type of definite integrals involving Bessel functions is provided by what are known as functions self-reciprocal in the Hankel transform

defined as follows.

$f(x)$ is self-reciprocal in the Hankel transform of order $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ i.e. $f \in R_\alpha$,

if $f(x) = \int_0^\infty (x/s)^{\frac{1}{2}\alpha} J_\alpha (2(sx)^{\frac{1}{2}}) f(s)\, ds$, where $(x/s)^{\frac{1}{2}\alpha} J_\alpha(\,.\,)$ stands for

$\prod_{i=1}^{n} (x_i/s_i)^{\frac{1}{2}\alpha_i} J_{\alpha_i} (2(s_i x_i)^{\frac{1}{2}})$, re $\alpha_i > -1$, and $f \in L$.

The following formal rules for functions self-reciprocal in the Hankel transform of order $\underset{\sim}{\nu}$ may be easily verified.

(i) If $f(p) \underset{n}{\subset} h(x)$ and if $h(x) \in R_\nu$, then $p^{1-(\nu)} f(1/p)=f(p)$; and if

$p^{(\nu)-1} f(p) \in R_\nu$, then $x^{(\nu)-1} h(1/x)=h(x)$ and conversely.

Also, if $x^{(\nu)-1} f(x) \in R_{(\nu)}$, then $x^{(\nu)+2(\alpha)} D_x^{(\alpha)} (f(x)/x) \in R_m$, where $m=(m_1,\ldots,m_n)$,

$m_i=\nu_i+2\alpha_i$ and $D_x^{(\alpha)}=\dfrac{\delta^{\alpha_1+\cdots+\alpha_n}}{\delta x_1^{\alpha_1}\ldots\delta x_n^{\alpha_n}}$; the converse also holds.

(ii) If $\psi(x) \underset{n}{\supset} p\, h(p)$, $x^{\gamma} h(x) / \underset{n}{\supset} f(p)$, then if $\psi(x) \in R_{(\mu)}$ then $x^{m-1} f(x) \in R_m$

where $m_i=2\gamma_i-2\alpha_i-\mu_i+2$. Also, if $x^{(\mu)-1} f(x) \in R_{(\mu)}$, then $\psi(x) \in R_{(m)}$.

In particular if $\psi(x) \in R_{(\mu)}$, re $(\mu) > -1$, $x^{2\beta-\mu-1} f(x) \in R_m$, $m_i=2\beta_i-\mu_i$, re $m_i > -1$,

and if $x^{(\mu)-1} f(x) \in R_{(\mu)}$, then $\psi(x) \in R_m$, provided that

$$(fp)/p=\prod_{i=1}^{n}\Gamma(\beta_i+1)\int_0^{\infty}\psi(s)/(p+s)^{(\beta)+1}\,ds,\quad (p+s)^{(\beta)+1}=\prod_{i=1}^{n}(p_i+s_i)^{\beta_i+1},$$

The following are certain examples of self-reciprocal functions derived from the above rules by using known operational relations in one and two variables.

Using operational representations for

$$(2x)^{\frac{1}{2}(\mu+\nu)}e^{-\frac{1}{2}x}D_{-(\mu+\nu+1)}((2x)^{\frac{1}{2}}),\ x^{m-\frac{1}{2}}/(1+x)^{n+\frac{1}{2}}\ (x+x^{2})^{\nu-\frac{1}{2}},$$

respectively, we have the following self-reciprocal functions in Hankel transforms of order indicated with each example.

(i) $x^{\frac{1}{2}(\mu+\nu)}e^{\frac{1}{2}x}D_{-2\alpha}((2x)^{\frac{1}{2}})\in R_{\alpha},\ \alpha=\frac{1}{2}(\mu+\nu+1);$

(ii) $x^{\frac{1}{2}(3n-m-2)}e^{\frac{1}{2}x}W_{-\frac{1}{2}(n+m),\,-\frac{1}{2}(n-m)}(x)\in R_{2n-m-\frac{1}{2}};$

(iii) $x^{2\nu-\frac{1}{2}}e^{\frac{1}{2}x}K_{\nu}(\frac{1}{2}x)\in R_{\alpha}$, where $\alpha=3\nu-\frac{1}{2}$, re $\nu\geqslant 0$.

Again, from the operational representations for $(pq)^{\frac{1}{2}}/(p^{\frac{1}{2}}+q^{\frac{1}{2}})^{n+1}$ and $x^{\nu}/\Gamma(\nu+1)\,J_{2\nu}(2(xy)^{\frac{1}{2}})$ we have

(iv) $(xy)^{\frac{1}{2}n}/(x+y)^{\frac{1}{2}(n+1)}\in R_{\alpha,\alpha},\ \alpha=\frac{1}{2}(n+1);$

(v) $(xy)^{\frac{1}{2}n}/(x^{\frac{1}{2}}+y^{\frac{1}{2}})^{n+1}\in R_{\alpha,\alpha},\ \alpha=\frac{1}{2}(n+1);$

(vi) $x^{\nu}J_{2\nu}(2(xy)^{\frac{1}{2}})\in R_{3\nu,\nu}$, re $\nu>1$.

6. **Summation of series.** In the following we use operational calculus to derive certain summation of series involving special functions.

(i) From the known expansion formula for the Laguerre polynomial $L_n^{\alpha}(x)$ and the relations for $L_n^{\pm\frac{1}{2}}(x)$ in terms of the parabolic cylinder functions $D_{2n}(x)$ and $D_{2n+1}(x)$ it follows that

$$1/n!\cdot 2^{-1/6(4k+1)}\cdot(-1)^n\, 2^{-1/6(6n+2k-1)}\cdot 1/p^k q^{n-\frac{1}{2}}\cdot e^{-\frac{1}{4}/pq}\, D_{2n}(1/(pq)^{\frac{1}{2}})$$

$$=\sum_{r=0}^{n}(-1)^r/r!\binom{n-\frac{1}{2}}{n-r}(2p)^{-k-r}\, q^{-n-r+\frac{1}{2}}\, e^{-\frac{1}{2}/pq}.$$

Interpreting both sides by the operational representations for

$p^{-k}\, q^{\frac{1}{2}-m} e^{-1/4pq}\, D_{2m}(1/(pq)^{\frac{1}{2}})$ and $p^{-m} q^{-n}\, e^{-1/pq}$ (See (3), (4))

we obtain

$$1/n!\; x^n J_{k,-\frac{1}{2}}(3x)=\sum_{r=0}^{n}(-1)^r/r!\binom{n-\frac{1}{2}}{n-r}x^r\, J_{k+r,\,n+r-\frac{1}{2}}(3x).$$

Similarly

$$1/n!\; x^n\, J_{\frac{1}{2}(2k+1),\,\frac{1}{2}}(3x)=\sum_{r=0}^{n}(-1)^r/r!\begin{pmatrix}n-\frac{1}{2}\\ n-r\end{pmatrix}x^r\, J_{k+r+\frac{1}{2},\,n+r+\frac{1}{2}}(3x),$$

where $J_{m,n}(\cdot)$ is the Bessel function of 'troisme' order.

A similar analysis with the expansion

$$M_{k,m}(x)=x^{m+\frac{1}{2}}e^{-\frac{1}{2}x}\sum_{r=0}^{\infty}(\tfrac{1}{2}+m-k)_r/(2m+1)_r\, r!\cdot x^r$$ leads to the expansion

$$x^{a-m}\, J_{a+m,2m}(3x)=$$

$$\Gamma(a+m+1) / \Gamma(2m+1) \sum_{r=0}^{\infty} (m-a)_r / r!\, (2m+1)_r, x^r J_{a+m+r,\, a+m+r} (3x)$$

re $(m+a) > -1$, $| x | < 1$.

(ii) Since $p^{-m} q^{-n} e^{-(a+b) / pq} = \sum_{r=0}^{\infty} (-1)^r / r!\, b^r p^{-m-r} q^{-n-r} e^{-a/pq}$,

it follows that

$$J_{m,n} (3x(a+b)^{\frac{1}{3}}) = (1+b/a)^{1/3(m+n)} \sum_{r=0}^{\infty} (-1)^r / r!\, (b/a^{2/3})^r ,x.^r J_{m+r,\, n+r} (3x\, a^{1/3})$$

(iii) We have the operational representation

$$x^a y^b L_m^{a,b} (x,y) \supset\supset \Gamma(m+a+1)\, \Gamma(m+b+1) / (m!)^2 p^{-a} q^{-b} (1-1/p-1/q)^m$$

where $L_m^{a,b}(\,.\,)$ is the general and Laguerre polynomial

$$\Gamma(a+1)\, \Gamma(b+1)\, (m!)^2 / \Gamma(m+a+1)\, \Gamma(m+b+1)\, x^a y^b\; L_m^{a,b} (x,y) =$$

$$x^a y^b \psi_2 (-m, a+1, b+1, x, y) = \sum_{\mu,\, \nu} (-m, \mu+\nu) / (a+1, \mu)\, (b+1, \nu)\, .x^{a+\mu} y^{b+\nu} / \mu!\, \nu!$$

$\psi_2(\,.\,)$ being the confluent hypergeometric function of two variables.

From the known operational representation for $x^a L_n^a (x)$, $L_n^a(\,.\,)$, Laguerre polynomial,

we have $x^a y^b L^a (x) L^b (y) \supset\supset$

$$\Gamma(n+a+1)\, \Gamma(n+b+1) / (n!)^2 p^{-a} q^{-b} \sum_{r=0}^{n} \binom{n}{r} \left(1-\frac{1}{p}-\frac{1}{q}\right)^r (pq)^{-n+\gamma};$$

from which $(xy)^{-n} L_n^a (x) L_n^b (y) = \sum_{r=o}^{n} r ! / n ! (n-r) ! (xy)^{-r} \; L_r^{a+n-r,\, b+n-r} (x y).$

Stnce $p^{-a} q^{-b} (1-1/p-1/q)^{n} = \sum_{r=o}^{n} (-1)^r \binom{n}{r} p^{-(a+r)} (1-1/p)^{n-r} q^{-(b+r)} (1-1/q)^{n-r},$

it follows from the known operational relation for $x^{\nu-\frac{1}{2}} e^{\frac{1}{2}x} M_{n+\nu+\frac{1}{2},\nu} (x)$, re $\nu > -\frac{1}{2}$,

$$(n!)^2 / \Gamma (n+a+1) \Gamma (n+b+1) x^{\frac{1}{2}(a+1)} y^{\frac{1}{2}(b+1)} e^{-\frac{1}{2}(x+y)} L_n^{a,\, b} (x,y)$$

$$= \sum_{r=o}^{n} (-1)^r \binom{n}{r} / \Gamma (a+r+1) \Gamma (b+r+1) (xy)^{\frac{1}{2}r} M_{\frac{1}{2}(k+a),\, \frac{1}{2}(r+a)} (x) \; M_{\frac{1}{2}(k+b),\, \frac{1}{2}(r+b)} (y)$$

where $k=2n-r+1$.

Similar formulae can be multiplied by using operational calculus of one and more variabtes.

References

1. Chakravarty N. K. On the convergence of Laplace Integral— Jour. Pure Math., 1,1981,1—7.

2. Erdelyi et al (a) Tables of Integral transforms, Vols I,II (1954) (Mcgraw Hill)
   (b) Higher Transcendental Functions, Vols. I—III, (1953,1955) (Mcgraw Hill)

3. Voelker, D. and Doetach, G. Die Zweidimensionale Laplace Transformation (1950) (Verlag Birkhauser Basle).

4.	Ditkin, V.A. and Prudnikov, A.P.	Operational Calculus in two variables and applications (1962) (Pergamon).
5.	Poli, L. and Delerue, P.	Le Calcul Symbolique a deux variables et ses applications Fasc CXXVII, (1954), Memorial des Sci. Math.

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On an integral equation connected with a differential system

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1. **Introduction.** The second order formally self-adjoint matrix differential expression under consideration is

$$lU=-D^2U+QU, \quad a\leqslant x\leqslant b \tag{1.1}$$

where i) $D=d/dx$, $U=(u(x).\ v(x))^T$ and $Q=Q(x)=\begin{pmatrix} p & r \\ r & q \end{pmatrix}$

and ii) the elements p, q, r are real and summable on every finite interval $[a,b]$, $b>a$.

Let H denote the basic Hilbert space $H=L_2\ [a,b)$ (where $a=-\infty$, $b=\infty$ may be allowed) and $\mathfrak{D}$ the set of all $U=(u,v)^T \in H$ such that U' exists and is absolutely continuous in each finite interval $[a,b]$, $b>a$ and $lU \in L_2\ [a,b)$.

Denote by $\mathfrak{D}_A$ the set of all vectors $U \in H$ which satisfies the conditions

$$a_{j1}\ u(0)+a_{j2}\ u'(0)+a_{j3}\ v(0)+a_{j4}\ v'(0)=0,\ j=1,2 \tag{1.2}$$

where a_{ij} are real numbers, $A=(a_{j1},\ a_{j2},\ a_{j3},\ a_{j4})$ is fixed, the a_{ij} satisfy $a_{11}\ a_{22}+a_{13}\ a_{24}-a_{12}\ a_{21}-a_{14}\ a_{23}=0$ (1.3)
and rank $(a_{ij})=2$.

Define the operator L_A as follows :

for $U \in \mathfrak{D}_A$, let $L_A\ U = lU$; so that L_A is the differential operator generated by the second order matrix differential expression lU and the boundary conditions (1.2) ; the domain of definition of L_A is $\mathfrak{D}_A$.

Let us choose the boundary condition at x=b as

$$B_{j1}\ u(b)+B_{j2}\ u'(b)+B_{j3}\ v(b)+B_{j4}\ v'(b)=0 \tag{1.4}$$

where $B=(B_{j1}, B_{j2}, B_{j3}, B_{j4})$ is fixed, B_{ij} real and satisfies

$$B_{11}\ B_{22}+B_{13}\ B_{24}-B_{12}\ B_{21}-B_{14}\ B_{23}=0 \tag{1.5}$$

and rank $(B_{ij}) = 2$.

If $\phi_j\ (x)=(x_j,\ y_j)^T$, j=1. 2, be the vectors taking initial values $x_j=a_{j2}$, $y_j = a_{j4}$, $x'_j = -a_{j1}$, $y'_j = -a_{j3}$, j=1,2, at x = 0, then (1.2) and (1,3) reduce to the simpler forms

$[U\ \phi_j]_o = .$ and $[\phi_1\ \phi_2]_o = 0$ where $[0]_o$ is the value of the bilinear concomitant of the two vectors $c_j=(c_{1j},\ c_{2j})^T$ (functions of x) defined by

$$[c_1,\ c_2]= \begin{vmatrix} c_{11} & c_{12} \\ c'_{11} & c'_{12} \end{vmatrix} + \begin{vmatrix} c_{21} & c_{22} \\ c'_{21} & c'_{22} \end{vmatrix} \qquad \text{at } x = 0.$$

A similar representation also holds for (1.4) and (1.5)

In view of the conditions (1.3) and (1 5) L_A (or a possible extension of it) is selfadjoint. (See Naimark (1), p. 73 ; Chakravarty (2), p. 138-39).

When Q = 0 we obtain similary a self-adjoint differential operator L_A^F to be called the Fourier differential operator generated by the second order matrix differential expression $l^F U$ (i.e. the differential expression (1.1) with Q=0) and the boundary conditions (1.2), (1.3) at x = 0, (1.4), (1.5) at x = b.

In a paper (See(3)) Roy Paladhi obtains for the system

$$lU = \lambda U,\ 0 \leqslant x < \infty\ , \tag{1.6}$$

where λ is real or complex, the integral equation

$$\phi(x,\lambda) = c\,(x,\lambda) + \int_0^x K(x,t)\,\phi(t,\lambda)\,dt$$

with inverse $c(x,\lambda) = \phi(x,\lambda) - \int_0^x M\,(x,t)\,c\,(t,\lambda)\,dt$

$c\,(x,\lambda)$ having linearly independent column vectors

$c_j\,(x,\lambda) = (a_{j2}\, \cos \lambda^{\frac{1}{2}}\, x + a_{j1} \sin \lambda^{\frac{1}{2}} x,\ a_{j4} \cos \lambda^{\frac{1}{2}} x + a_{j3} \sin \lambda^{\frac{1}{2}} x)$,

$j = 1,2$; $\phi_j(x,\lambda)$ satisfies the prescribed boundary conditions at $x = o$, if

$$a_{j1}\, a_{k2} + a_{j3}\, a_{k4} = 0,\ j, k = 1, 2. \tag{A}$$

a necessary and sufficient condition for the validity being that $K(x,t)$, $M(x,t)$ satisfy a set of conditions elaborated in his paper under reference.

In the present paper we derive a similar but more general integral equation by a different method. The condition (A) is replaced by a wider one. The integral equation so derived, is finally utilised to derive a theorem on the uniqueness of the inverse problem, i.e. the problem of reconstruction of the differential system from given spectral data. This incidentally corrects a wrong conclusion arrived at by Roy Paladhi in his paper.

2. **The integral equations.** Let α_{ij}, $i,j = 1,2$ be a set of real constants such that

$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{21} & \alpha_{23} \end{pmatrix}$ is singular and $\beta = \begin{pmatrix} \alpha_{12} & \alpha_{14} \\ \alpha_{22} & \alpha_{24} \end{pmatrix}$ is non-singular ;

and α_{ij} are connected with a_{ij} by means of the relation

$$a_{j1}\, \alpha_{k2} + a_{j3}\, \alpha_{k4} = a_{j2}\, \alpha_{k1} + a_{j4}\, \alpha_{k3} \tag{1,7}$$

where $k = 1$, $j = 1,2$; $k = 2$, $j = 1,2$.

A matrix $\phi(x,\lambda)$ is said to be the solution of (1.6) if each of the linearly independent column vectors $(\phi(\ ))_k$, $k = 1,2$, of $\phi(x, \lambda)$ satisfies (1.6) ; $\phi(x,\lambda)$ satisfies the boundary conditions (1.2), (1.3) at $x = 0$, if each $(\phi(x, \lambda))_k$ does so, Put $$(\phi^{F}(x,\lambda))_k = \begin{pmatrix} \alpha_{k2}\cos\lambda^{\frac{1}{2}}x - \alpha_{k1}\lambda^{-\frac{1}{2}}\sin\lambda^{\frac{1}{2}}x \\ \alpha_{k1}\cos\lambda^{\frac{1}{2}}x - \alpha_{k3}\lambda^{-\frac{1}{2}}\sin\lambda^{\frac{1}{2}}x \end{pmatrix} \tag{1.8}$$

where λ is real or complex.

The matrix $\phi^{F}(x, \lambda)$ with two linearly independent column vectors $(\phi\,(.))_k$ defined in (1.8) $(k = 1,2)$ satisfies the Fourier system

$$l^{F}\,U = \lambda U \qquad (1\ 9)$$

corresponding to (1.6). In view of (1.7), $(\phi^{F}\,(.))_k{}'$ $k = 1,2$, and therefore $\phi^{F}(x,\lambda)$ satisfies the boundary conditions (1.2), (1.3).

We assume that a_{ij}, B_{ij}, α_{ij} are all independent of the parameter λ.

Let there exist a non-singular, symmetric 2×2 matrix $K(x,t) = (K_{ij}(x,t))$ $K_{ij}=K_{ji}$, defined for $0 \leqslant t \leqslant x$, $K(x,t) = 0$ for $t > x$ or for $t < 0$, having continuous partial derivatives upto the second order with respect to x, t such that the matrix integral equation

$$\phi(x,\lambda) = \phi^{F}(x,\lambda) + \int_{0}^{x} K(x,t)\,\phi^{F}(t,\lambda)\,dt \tag{1.10}$$

holds We investigate the conditions to be satisfied dy $K(x,t)$ in order that $\phi(x,\lambda)$ defined by (1.10) shall satisfy (1.6) and the boundary conditions (1.2), (1.3).

Theorem 1 1. If $Q(x)$ is continuously differentiable in $0 \leqslant x \leqslant a$, a finite, then a necessary and sufficient condition that the matrix $\phi(x,\lambda)$ uniquely defined by (1.10) is a unique solution of (1.6) satisfying the boundary conditions (1.2), (1.3), is that

(i) $K(x,t)$ is the unique solution of the partial differential equation

$$K_{xx} - K_{tt} = K(x)\,K(x,t)$$

with

(ii) $K(x,x) = \frac{1}{2}\int_{0}^{x} Q(s)\,ds$; $K(0,0) = 0$;

(iii) $K(x,)0 \neq 0$; $K_t(x,t)\Big|_{t=0} = 0.$

The solution $\phi(x,\lambda)$ so obtained is an entire function of λ.

Our method is similar to Rofe-Beketov (4) (for systems of equations) and Agranovich & Marchenko (5) and Faddeyev (6) for the Strum-Liouville equations. Our results are more general. We therefore outline the method emphasisig only those parts where we considerably differ.

It follows from $\int_o^x \sin \lambda^{\frac{1}{2}}(x-y)\, Q(y)\, \phi(y)\, dy$ by replacing $Q(t)\,\phi(t)$ by

$D^2\,\phi(t) + \lambda\,\phi(t)$, $\phi(t) = \phi(t,\lambda)$, and then integrating by parts twice, that

$$\phi(x,\lambda) = \phi^F(x,\lambda) + \lambda^{-\frac{1}{2}} \int_o^x \sin \lambda^{\frac{1}{2}}(x-y)\, Q(y)\, \phi(y)\, dy \tag{1.11}$$

$\phi(x,\lambda)$ (i.e. its column vectors $(\phi(.))_j$) satisfy (1.6) and the boundary conditions (1.2), (1.3) at $x=o$.

Again since $Q(x)$ is continuous, it follows by iteration that there exists a unique solution $\phi(x,\lambda)$ of (1.11) which satisfies (1.6) and the boundary conditions (1.2), (1.3) at $x=o$; also $\phi(x,\lambda)$ is an entire function of λ. (See Titchmarsh (7), p. 351),

Put $\lambda=\mu^2$. Then it follows formally from (1.10) and (1.11) that

$$\int_o^x K(x,t)\, \phi^F(t,\lambda)\, dt = \mu^{-1} \int_o^x \sin \mu(x-s)\, Q(s)\, ds\, \Big(\phi^F(s\,\mu) + \int_o^s K(s,u)\, \phi(u,\mu)\, du\Big)$$

$$= I_1 + I_2, \text{ say} \tag{1.12}$$

(by using $\sin \mu(x-s) \cos \mu s \,/\, \mu = \frac{1}{2} \int_{25-x}^{x} \cos \mu u\, du$ and

$\sin \mu(x-s)\ \sin \mu s \,/\, \mu = -\frac{1}{2} \int_{25-x}^{x} \sin \mu t\, dt$)

Here $I_1 = \frac{1}{2} \int_0^x Q(s)\, ds \int_{25-x}^{x} \phi^F (t,\mu)\, dt$ and

$$I_2 = \frac{1}{2} \int_0^x Q(s)\, ds \int_0^s K(s,u)\, du \int_{4-x+s}^{4+x-s} \phi^F (t,\mu)\, dt.$$

Changing the order of integration which is easily justifiable,

$$I_1 = \frac{1}{2} \int_0^x dt \, \Big(\int_0^{\frac{1}{2}(x+t)} Q(s)\, ds \Big) \, \phi^F (t,\mu).$$

In I_2 we change the order of integration (also easily justifiable) in such a way that t varies from 0 to x ($t \leqslant x$) ; then fixing t the integral is taken over the region in the s-u plane bounded by $s=u$, $s-u=x-t$, $s+u=x+t$ and $u=0$. This in turn leads to the sum of integrations over two regions in the s-u plane :

(i) region bounded by $t+s-x \leqslant u \leqslant t-s+x$, $\frac{1}{2}(x+t) \leqslant s \leqslant x$ and

(ii) that by $s+t-x \leqslant u \leqslant s$, $o \leqslant s \leqslant \frac{1}{2}(x+t)$. Then

$$I_2 = \frac{1}{2} \int_0^x dt \, \Big(\int_0^{\frac{1}{2}(t+x)} Q(s)\, ds \Big) \Big(\int_{t-x+s}^{s} K(s,u)\, du \Big) \phi^F (t,\mu) -$$

$$-\frac{1}{2} \int_0^x dt \, \Big(\int_x^{\frac{1}{2}(x+t)} Q(s)\, ds \Big) \Big(\int_{t-x+s}^{t+x-s} K(s,u)\, du \Big) \phi^F (t,\mu)$$

We substitute for I_1, I_2 in (1.12) and (1.10) follows from (1.11) by putting

$$K(x,t) = \frac{1}{2} \int_o^{\frac{1}{2}(x+t)} Q(s)\, ds + \frac{1}{2} \int_o^{\frac{1}{2}(x+t)} Q(s)\, ds \int_{t-x+s}^{s} K(s,u)\, du - \frac{1}{2} \int_x^{\frac{1}{2}(t+x)} Q(s)\, de \int_{t-x+s}^{t+x-s} K(s,u)\, du \qquad (1.13)$$

It follows by iteration that (1.13) has a unique solution for $K(x,t)$ and if

$$\tau (x) = \int_o^x \| Q(s) \|\, ds, \quad \tau_1 (x) = \int_o^x t \, \| Q(t) \|\, dt, \text{ where.}$$

$\| A \| = \max\limits_{1 \leqslant i,j \leqslant 2} | a_{ij} |$, $A = (a_{ij})$, then

$$\| K(x,t) \| \leqslant \tfrac{1}{2} \tau'(\tfrac{1}{2}(x+t)) \exp \tau_1 (x) \tag{1.15}$$

(See Rofe-Beketov (4) ; Agranovich and Marchenko (5)).

We now seek for the properties of $K (x,t)$.

The property (ii) is obvious from (1.13).

To prove (i), differentiate (1.13) twice with respect to x and t respectively, subtract and utilise the relations

$K_x (s,t+x-s) = K_t (s,t+x-s)$ and $K_x (s,t-x+s) = -K_t (s,t-x+s)$.

It is well-known that the solution $K(x,t)$ of the Cauchy problem (i) with initial conditions $K(x,t)\Big|_{t=0} = f(x)$ and $K_t (x,t)\Big|_{t=0} = h(x)$, where $f(x)$ is twice differentiable and $g(x) = \int\limits_0^x h(x)\, dx$, $h(x)$, an integral of an absolutely continuous function, found uniquely by the Riemann method, is

$$K(x,t) = \tfrac{1}{2} (f(x+t) + f(x-t) + g(x+t) - g(x-t)) + \tfrac{1}{2} \int\limits_{x-t}^{x+t} (W(x,t,s)\, f(s) - T(x,t,s)\, g(s))\, ds \tag{1.16}$$

W (.), T(.) being the Riemann matrices for the given problem. (See Chakravarty and Roy Paladhi (8)). From this it follows that $K(x,t)$ is non-null, if and only if $K(x,0)$ and $K_t (x,t)\Big|_{t=0}$ are not zero simultaneously. (Roy Paladhi's condition (3.10), p.175, as given in his paper (3) under reference thus appears wrong.) Differentiating (1·10) twice with respect to x, replace $K_{xx} (x,t)$ by $K_{tt} (x,t)$ as given by the property (i) and then integrate by parts twice the integral involving $K_{tt} (x,t)$. Then since $\phi(x,\lambda)$ and $\phi^F (x,\lambda)$ satisfy (1.6) and (1.9) respectively, it follows that

$$K_t(x,t)\Big|_{t=0}(\phi^F(x,\lambda)-\phi^F(0,\lambda))+K(x,0)\,\phi^{F'}(0,\lambda)=0 \tag{1.17}$$

where we have utilised the result

$$K_t(x,t)\Big|_{t=x}=(K_{t-x}(x,t-x)\Big|_{t-x=0}=K_t(x,t)\Big|_{t=0}.$$

In (1.17) put $\lambda^{\frac{1}{2}}=2n\pi\,/\,x$ and $n\pi\,/\,x$ separately, where n is any positive integer, so as to obtain

$K(x.0)\,\alpha=0$ and $K_t(x,t)\,'_{t=0}\,\beta=0$, where α, β are matrices defined in (1.7). Since α is singular, there exists $K(x,0)\neq 0$. Again, since β is non-singular, $K_t(x,t)\Big|_{t=0}=0$. The property (iii) therefore follows.

Conversely, suppose that K(x,t) uniquely determined by the properties (i) — (iii) (as can be obtained by the Riemann method stated before) is given. Then following Titchmarsh (7), p.353-354, we can show that $\phi(x,\lambda)$ satisfies (1.10) and the boundary conditions (1.2), (1.3) at x=0. The theorem is therefore completely proved.

It follows from (1·10) by the theory of the Volterra Integral equations that there exists a symmetric, non-singular matrix M(x,t), the reciprocal of K(x,t), such that

$$\phi^F(x,\lambda)=\phi(x,\lambda)-\int_0^x M(x,t)\,\phi(t,\lambda)\,dt \tag{1.18}$$

where $\phi(x,\lambda)$ is uniquely determined.

The following theorem is now established.

Theorem 1 2. If Q(x) is continuously differentiable in $0\leqslant t\leqslant x\leqslant a$, a finite, a necessary and sufficient condition that $\phi(x,\lambda)$ given by (1 18), should satisfy the differential system (1 6) with boundary conditions (1 2), (1.3), is that M(x,t) should satisfy the conditions

(i) $M_{xx} - M_{tt} = -Q(t)M\,(x,t)$;

(ii) $M(x,x) = \frac{1}{2}\int_0^x Q(s)\,ds, \; M(0,0) = 0$;

(iii) $M(x,0) \neq 0, \; M_t \Big|_{t=0} = 0$;

(iv) $M(x,t) = 0$, $t > x$ or $t < 0$ and $M(x,t)$ has continuous partial derivatives upto the order two with respect to x, t.

We prove the properties (i) — (iv) from those of $K(x,t)$ of theorem 1.1. The theorem is then a consequence of the theory of Volterra Integral equations. It follows from (1.10), (1·18) and the uniqueness of the ϕ^F — Fourier transform that

$$K(x,t) = M(x,t) + \int_t^x M(x,u)K(u,t)\,du \tag{1,19}$$

Properties (ii) and (iv) follow from (1·19) and the corresponding known properties of $K(x,t)$. By differentiating (1.19) it follows that $M_t \Big|_{t=0} = 0$.

Property (i) follows from (1.19) in the same way as in Rofe-Beketov (4). Solving the partial differential (i) for $M(x,t)$ by the Riemann method it follows that $M(x,0) \neq 0$, since $M(x,t)$ is non-trivial and $M_t \Big|_{t=0} t = 0$. The property (iii) therefore holds for $M(x,t)$.

Since $M(x,t)$ is uniquely determined from the conditions (i) — (iv), the converse follows by the theory of Volterra equations and the theorem 1.1. The theorem is therefore completely established.

We establish the following inequality for $M(x,t)$ similar to that for $K(x,t)$ as derived in (1.15). $\tau(x)$ and $\tau_1(x)$ are the same as in (1.14).

In view of (1.15) it follows from (1.19) that

$$\| M(x,t) \| \, \tau^{-1}(\tfrac{1}{2}(x+t) \leqslant C + C\tau^{-1}(t) \int_t^x \| M(x,u) \| \; \tau(\tfrac{1}{2}(t+u))\,du,$$

where $C=\frac{1}{2}\exp(\tau_1(x))$ and for $t < x$. $\frac{1}{2}(t+x) > t$, which implies that $\tau^{-1}(\frac{1}{2}(t+x)) < \tau^{-1}(t)$. Thus

$$\| M(x,t) \| \; \tau^{-1}(\tfrac{1}{2}(x+t)) \leqslant C + \int_t^x \| M(x,u) \| \, \tau^{-1}(\tfrac{1}{2}(x+u)) \times (C\tau^{-1}(t)\, \tau^2(\tfrac{1}{2}(t+u))\, dt.$$

Hence by the Gronwall inequality

$$\| M(x,t) \| \, \tau^{-1}(\tfrac{1}{2}(x+t)) \leqslant C \exp \int_t^x \| g \| \; du \tag{1.20}$$

$$\text{where} \int_t^x \| g \| \, du \leqslant C\tau^{-1}(t)\, \tau(\tfrac{1}{2}(t+x)) \int_t^x \tau(\tfrac{1}{2}(t+u))\, du. \tag{1.21}$$

$$\text{Since} \int_t^x \tau(\tfrac{1}{2}(t+u))\, du = \int_t^x du \int_0^{\frac{1}{2}(t+u)} \| Q \| \; dv \leqslant \int_t^x du \int_0^u \| Q \| \; dv$$

$$= \int_t^x v \, \| Q \| \, dv \leqslant \tau_1(x),$$ it follows from (1.20) and (1.21) that

$$\| M(x,t) \| \leqslant C\, \tau(x) \exp(C\tau^{-1}(t)\, \tau(x)\, \tau_1(x)) \tag{1.22}$$

2. A generalization of theorems 1.1, 1 2. The operator T defined by

$$Tg=f=g+\int_0^x K(x,t)\; g(t)\; dt \tag{2.1}$$

where f,g are 2×2 matrices with two linearly independent column vectors and $K(x,t)$ is the karnel, continuous in the variables x,t, which vanishes for $t>x$ and $t<0$; $K(x,t)=K_{ij}(x,t))$, $K_{ij}=K_{ji}$, is called the Volterra Operator. If g is continuous, so is f. Also from the Schwarz inequality and the inequality

$(a+b)^2 \leqslant 2(a^2+t^2)$, it follows that if $g \in L_2 [\, 0,\infty)$, then $f \in L_2 [\, 0,\infty)$.

By the theory of Volterra equations it follows from (2.1) that

$$g=f - \int_0^x M(x,t)\, f(t)\, dt \tag*{2.2)}$$

where the kernal $M(x,t)$, continuous in x,t, vanishes for $t>x$ and for $t<0$. (2.2) defines the operator T^{-1}, inverse to T defined by (2.1). The mapping T or T^{-1}, is one to one and it is easy to prove that the Volterra operators form a group under multiplication.

Along with the self-adjoint differential operator L_A generated by the differential system (1.6) with boundary conditions (1.2), (1.3) let us consider a second self-adjoint differential operator L_B generated by

$$l_1\, U=\lambda\, U, \quad 0\leqslant x<\infty \tag*{(2.3)}$$

where l_1 is the matrix differential expression (1.1) with Q replaced by $Q_1=\begin{pmatrix} p_1 & r_1 \\ r_1 & q_1 \end{pmatrix}$, Q_1 satisfying conditions similar to Q ; and the boundary conditions at $x=0$ similar to those of (1.2) and (1.3) with a_{ij} replaced by b_{ij}, say, b_{ij} being related to α_{ij} by

$$b_{j1}\, \alpha_{k2} + b_{j3}\, \alpha_{k4}=b_{j2}\, \alpha_{k1} + b_{j4}\, \alpha_{k3}, \text{ where } k=1, j=1,2;\ k=2, j=1,2. \tag*{(2.4)}$$

As in theorems (1.1 and 1.2, a necessary and sufficient condition that

$$\phi_1 (x,\lambda) =\overset{F}{\phi} (x,\lambda) + \int_0^x K_1 (x,t)\, \overset{F}{\phi} (t,\lambda)\, dt$$

with inverse $\overset{F}{\phi} (x,\lambda) = \phi_1 (x,\lambda) - \displaystyle\int_0^x M_1 (x,t)\, \overset{F}{\phi} (t,\lambda)\, dt$ (2.5) should satisfy (i.e. the vectors $(\phi_1(\,.\,))_k$, $k=1,2$, satisfy) the system (2.3) and the boundary conditions as above prescribed, is that the kernels $K_1(\,.\,)$ and $M_1(\,.\,)$ should satisfy relations similar to those of $K(\,.\,)$ and $M(\,.\,)$ in theorems 1·1 and 1.2.

Let $\phi(x,\lambda)$ be the solution of (1.6) and $\phi_1(x,\lambda)$ that of (2,1). Let $K_1(.)$ and $M_1(.)$ be associated with $\phi_1(x,\lambda)$ in the same way as $K(.)$, $M(.)$ with $\phi(x,\lambda)$.

$$\text{Put } \phi(x,\lambda) = \phi_1(x,\lambda) + \int_0^x K_2(x,t)\,\phi(t,\lambda)\,dt \tag{2.6}$$

$$\text{and } \phi_1(x,\lambda) = \phi(x,\lambda) - \int_0^t M_2(x,t)\,\phi t,\lambda)\,dt \tag{2.7}$$

Then the following theorem holds.

Theorem 2.1. Let Q, Q_1 be continuously differentiable in $0 \leqslant x \leqslant a$, a, finite. Also, let $\phi(x,\lambda)$, $\phi_1(x,\lambda)$, defined by (2.6), (2.7), be the solutions of the differential systems (1.1), (2.3). Then $K_2(x,t)$, $M_2(x,t)$ satisfy the following conditions.

(i) $K_2(x,t)$, $M_2(x,t)$ are symmetric having continuous partial derivatives with respect to x,t upto the order two;

(ii) $K_2(x,t)$, $M_2(x,t) = 0$ for $t>x$ and for $t<0$,

(iii) $K_2(x,x) = M_2(x,x) = \frac{1}{2}\int_0^x (Q(x)-Q_1(x))\,dx$;

(iv) $\delta/\delta t\, K_2(x,t)\Big|_{t=0} = 0$, $\delta/\delta t\, M_2(x,t)\Big|_{t=0} = 0$, but $K(x,0) \neq 0$, $M(x,0) \neq 0$;

(v) $\triangle_{xt} K_2(x,t) = (Q(x) - Q_1(t))\,K_2(x,t)$,

$\triangle_{xt} M_2(x,t) = (Q_1(x) - Q(t))\,M_2(x,t)$, where $\triangle_{xt} = \delta^2/\delta x^2 - \delta^2/\delta t^2$.

Conversely, let $K_2(x,t)$, $M_2(x,t)$ satisfy the conditions (i) — (v) and α_{ij} conficients in $\phi^F(.)$ be given. Then if $\phi_1(x,\lambda)$ be the solution of (2.3) satisfying the boundary conditions of type (1.2), (1.3) involving b_{ij} in place of a_{ij} and the relation (2.4), then $\phi(x,\lambda)$ is the solution of (1.6) satisfying the boundary conditions (1.2), (1.3) if α_{ij} satisfy (1.7). The theorem remains true if ϕ, ϕ_1 are the linearly independent solutions of the same equation (1.6) but satisfy different boundary conditions (1.2), (1.3), (1.7) at $x=0$.

The theorem is true if $\phi(x,\lambda)$ and $\phi_1(x,\lambda)$ are respectively the L_2 solutions of (1.6) and $(2.3)_2$ or if ϕ, ϕ_1 are the linearly independent L_2 solutions of the same equation (1 6), or if ϕ, ϕ_1 are the solutions of two different equations (1-6), (2.3) saftisfying the same boundary conditions at $x=0$.

From (1.18) and (2.5) we have

$T_1\phi(x,\lambda) = \phi^F(x,\lambda)$ and $T_2\phi_1(x,\lambda) = \phi^F(x,\lambda)$ i.e. $T_2^{-1}\phi^F(x,\lambda) = \phi(x,\lambda)$

where T_1, T_2 are the Volterra matrix operators.

Hence $T_2^{-1}\, T_1\, \phi(x,\lambda) = \phi_1(x,\lambda)$ and $T_1^{-1}\, T_2\phi_1\,(x,\lambda) = \phi(x,\lambda)$.

Writing explicity we derive (2.6), (2.7), where

$$K_2(x,t) = K(x,t) - M_1(x,t) - \int_t^x K(x,s)\, M_1(s,t)\, ds \tag{2.8}$$

with a similar relation for $M_2(x,t)$.

From known conditions for $K(x.t)$, $M_1(x,t)$, the properties (i) (ii) and (iii) for $K_2(x,t)$ follow from (2.8). The first part of (iv) follows by differentiating (2.8) and making $t \to o$. The first part of (v) is established by a straightforward differentiation twice of the relation (2 8) with respect to x, t and integrating by parts as and when necessary in the intermediate steps as before. Since the solution of the Cauchy problem for $K_2(x,t)$ has to be non-null, it follows by solving the partial differential equation for $K_2(x,t)$ with $K_2(x,o) = f(x)$ and $\delta / \delta t\, K_2(x,t) \Big|_{t=0} = 0$ by the Riemann method that $f(x) \neq o$.

Properties of $M_3(x,t)$ follow similarly.

Conversely, let $K_2(x,t)$ satisfy the conditions (i) — (v) and α_{ij} etc. be defined as in the theorem. Then $K_2(x,t)$ is uniquely determind. Also proceeding as in Titchmarsh (7), p. 353-54, we can show that $\phi(x,\lambda)$ defined by (2.6) cxists when $\phi_1(x,\lambda)$ is the solution of (2.3), with prescribed boundary conditions at $x=0$. That $\phi(x,\lambda)$ satisfies the differential system (1.6) and the boundary conditions, follow from (2.6) by twice differentiating with respect to x, utilising the properties of $K_2(x,t)$ and integrating by parts in the intermediate steps after using the relation (v) for $K_2(x,t)$.

Similarly for the equation (2.7), reciprocal to (2.6).

Now let ϕ, ϕ_1 be solutions of (1.6) and (2 3), ϵL_2.

Put $\lambda^{\frac{1}{2}} = \mu = \sigma + it$, $t>o$, and consider the integral equation

$$\phi(x,\lambda) = \exp(i\mu x)\, I + \int_0^x K(x,t)\ \exp(i\,\mu\, t)\, dt \tag{2.9}$$

$$\text{and } I \exp\ (i\mu x) = \phi(x,\lambda) - \int_0^x M(x,t)\ \phi(t,\lambda)\ dt \tag{2.10}$$

where I is the 2×2 unit matrix ; with a similar pair of equations for ϕ_1.

As in Titchmarsh (9), P 119, the integral equation

$$\phi(x,\lambda) = \exp(i\,\mu\, x)\, I + \mu^{-1} \int_0^x \sin \mu(x-y)\, Q(y)\, \phi(y)\, dy \tag{2 11}$$

can be solved by iteration ; $\phi(x,\lambda) \epsilon L_2$ and as can be readily seen, that it satisfies the differential system (1.6). Hence as in theorems 1.1 and 1.2, $\phi(x,\lambda)$ can be presented in the form (2.9) with reciprocal equation given by (2.10). $K(x,t)$ and $M(x,t)$ (different from those in theorems (1.1) and (1.2) satisfy all the properties of the corresponding entities in the theorems 1.1 and 1.2. Hence the theorem follows as before when ϕ, $\phi_1 \epsilon L_2$. The remaining parts of the theorm are now evident. The theorem is therefore completely established.

A similar generalization for the sturm-Liouville equation occurs in Thurlow (10).

3. A theorem on the uniqueness of inverse problem. For the differential operator L_a (i.e. equation (1 6) with boundary conditions (1.2), (1.3)) Tiwary (11), (12), following Weyl's celebrated Math. Annalen (1910) paper, considered a symmetric, monotone non-decreasing (in the sense of operator) matrix $\varrho(\lambda)$, λ real, $\varrho(-\infty)<\infty$, $\varrho(0)=0$, having the following properties.

(i) $d\varrho(\lambda)$ is positive in the sense that the corresponding quadratic from is positive .

(ii) if $f \in L_2\,[0,b)$, $\varrho(\lambda)$ generates an isometric mapping of $L_2[0,b)$ onto $L^2_\varrho\,(-\infty,\infty)$, the Hilbert space of matrices square integrable on $(-\infty,\infty)$ with weight $d\varrho$, by means of the formulae

$$E(\lambda) = E(\lambda,f) = \int_0^b \phi^T(x,\lambda)\, f(x)\, dx, \quad f(x) = \int_{-\infty}^{\infty} \phi(x,\lambda)\, d\varrho\, E(\lambda),$$

where ϕ is the solution of (1.6) with boundary condions (1.2), (1.3) at $x=0$, with satisfaction of the Parseval relation

$$\| f \|^2 = \int_0^b | f |^2 dx = \int_{-\infty}^{\infty} E^T(\lambda)\, d\varrho(\lambda)\; E(\lambda).$$

The generalized spectral matrix has been conisdered by Rofe—Beketor (4) ; $\varrho(\lambda)$ being its special case.

The problem of reconstruction of the differential system when $\varrho(\lambda)$ with properties, called the spectral data ($\varrho(\lambda)$ is the spectral matrix), is given, is the inverse problem. When $\varrho(\lambda)$ is given, the inverse problem for the differential operator L_A was solved by Roy Paladhi (3), p.169-92. We obtain below a theorem on the uniqueness of the inverse problem when $\varrho(\lambda)$ is given.

Theorem 3.1. The spectral matrix corresponding to the differential operator L_A generated by the differential expression l in (1.1) and the boundary conditions (1.2), (1.3) completely determines l.

Let $\varrho(\lambda)$ be the spectral matrix of the differential operator L_A and $\sigma(\lambda)$ that of L_B generated by l_1 given by (2.3) and the boundary conditions similar to (1 2), (1.3) at $x=0$ with a_{ij} replaced by b_{ij}. If $\phi(x,\lambda)$ is the matrix of the linearly independent boundary condition vectors, i.e. vectors satisfying (1.6) and taking initial values at $x=0$, namely $\phi_j=(a_{j2}, a_{j4})^T$, $\phi_j'=-(a_{j1}, a_{j3})^T$, we have $E(u)=\int_0^a \phi^T(x,u)\, f(x)\, dx$ implies that

$$f(x) = \int_{-\infty}^{\infty} \phi(x,u)\, d\,\varrho(u)\, E(u).$$

A similar formula holds for the matrix ϕ_1 of the boundary condition vectors (linearly independent) of (2.3) generating the differential operator L_B.

The volterra operator

$$g(x)=Tf=f(x)+\int_0^x K(x,t)\, f(t)\, dt$$

which maps, in particular, $\phi(x,\lambda)$ onto $\phi_1(x,\lambda)$, may be written in the form

$T=E+K$, where E is the identity operator in our case and

$K=K(f)=\int_0^x K(x,t)\, f(t)\, dt$ is quasi-nilpotent. (See Kato (13), p.154, Ex 3 15).

If $\varrho(\lambda)=1/C\, \sigma(\lambda)$, then closely following Marchenko (14), p.56-58, we obtain

$$Tf=\lim_{N\to\infty} T\left(\int_{-N}^{N} \phi(x,u)\, d\,\varrho(u)\, E(u)=1/C \lim_{N\to\infty} \int_{-N}^{N} \phi_1(x,u)\, d\,\sigma(u)\, E(u)\right.$$

where the integral on the right converges in the metric of $L_2\,[0,a)$ and Tf belongs to this space. Also $E(\lambda) \in L^2_{\varrho}\,(-\infty,\infty)$ implies $E(\lambda) \in L^2_{\sigma}(-\infty,\infty)$. (Marchenko's idea relating to the weak boundedness of the operator (see (14), p.25-26) needed in the proof can be seen to hold also for our operator. The second result needed is that $\varrho(\lambda)$ generates an isometric mapping of $L_2[0,a)$ onto $L^2(-\infty,\infty)$).

Since $Tf \in L_2[0,a)$ we have by the Parseval relation

$$c\int_0^a |f|^2 dx = \int_{-\infty}^{\infty} E^T(u)\,d\sigma(u)\,E(u) = \int_0^a |g|^2\,dx.$$

By a suitable normalization we can choose $C=1$. Then it follows that T is an L_2-isometric operator, T^{-1} exists and T is unitary. Therefore the operator K is normal. Since a quasi-nilpotent normal operator is zero, we have $K=0$. Thus $K(x,t)=0$.

Hence $Tf=f$ which implies that $\phi(x,\lambda)=\phi_1(x,\lambda)$. Thus $Q(x)=Q_1(x)$. The theorem therefore follows.

It may be remarked that Roy Paladhi's conclusion relating to nonuniqueness of a differential system constructed from a given spectral data as obtained in his theorem 5,p.191 in (3) runs counter to the present theorem and as such appears mistaken. In fact in the notation of Roy Paladhi we obtain

$$\|f\|_{0,\infty} = \pi^{-1}\|F,\ d\varrho\|_{-\infty,\infty},\quad F_j = \int_0^{\infty} \phi_j^T f\,dx. \quad \text{Thus}$$

$\|\overline{F},d\varrho\| = \|,d\varrho\|\,F$, which leads to $\overline{F}=F$ and this in turn leads to $A(t)=1$, i.e. $(H)=\phi$ in his counter-example in § 10,p.190-191. This is what is to be expected from our theorem.

References :

1. Naimark, M.A. Linear differential operator, part II, 1968, Ungar, N. Y.
2. Chakravarty, N.K. Some problems in eigenfunction expansions (I), Quart. J. Math. (Oxford), (2), 1965, 16,135-150.

3. Roy Paladhi, Basudeb The inverse problem associated with a pair of second order differential equations, Proc. Lond. Math. Soc. ,1981,43,169-192.

4. Rofe-Beketov, F.S. Eigenfunction expansions for infinite systems of differential equations in the self-adjoint and non-self-adjoint cases, Mat. Sb., 1960, 51(93), 293-342. (Russian).

5. Agranovich, Z. S and Marchenko, V.A. The inverse problem of scattering theory, 1963, Gordon and Breach, N.Y.

6. Faddeyev, L. D. The inverse problem in the Quantum theory of scattering, Jour. Math. Physics, 1963, 4(1), 72-104.

7. Titchmarsh, E.C. Eigenfunction expansions associated with second order differential equations, Part II, 1958, Clarendon Press, Oxford.

8. Chakravarty, N K. and Roy Paladhi, Swapna A Cauchy type problem for a second order matrix differential operator, J. Pure Math., Cal. Univ., 1984, 4,17-31.

9. Titchmarsh, E.C. Eigenfunction expansions associated with second order differential equations, Part I, 1946, Clarendon Press, Oxford.

10. Thurlow, Christine A generalization of the inverse spectral theorem of Levitan and Gasymov, Proc. Roy. Soc. Edinburgh, 1979, 84A, 72-104.

11. Tiwari, S. On the theory of transforms associated with eigenvectors (I), J. Ind. Inst. Sci., 1977,99,501-524.

12. Tiwari, S. On eigenfunction expansions associated with differential equations, Ph.D thesis, Calcutta University, 1971.

13. Kato, Tosio Perturbation theory of linear operators, 1980, Springer-Verlag, N.Y.

14. Marchenko, V.A. Some questions in the theory of one dimensional linear differential operators of the second order I, Am. Math. Soc. Transl., (2), 1973,101, 1-104.

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On a class of entire functions defined by Dirichlet Series

MD. MUKLESUR RAHMAN

1. **Introduction :** An entire function f represented by a Dirichlet series is of the form

(1.1) $$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \; (s = \sigma + i\tau),$$ where it is tacitly assumed that

$$0 \leqslant \lambda_1 < \cdots\cdots\cdots < \lambda_n \to \infty \text{ as } n \to \infty \text{ and}$$

(1.2) $$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0\,.$$

It is known [1] that a necessary and sufficient condition for the series (1.1) satisfying (1 2) to be entire is that

(1.3) $$\lim_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty.$$

Let Ω denote the family of all complex-valued functions of the form (1.1) with the same sequence of integral exponents $\{\lambda_n\}$ where $\lambda_n! \; |a_n|$ is bounded. It can be verified easily that the elements of this set are all entire functions. We now define addition, scalar multiplication and multiplication in Ω by

$$f_1(s) + f_2(s) = \sum_{n=1}^{\infty} (a_n + b_n)\, e^{s\lambda_n}$$

$$\alpha f_1(s) = \sum_{n=1}^{\infty} \alpha\, a_n\, e^{s\lambda_n}, \quad \alpha \in C$$

$$f_1(s) \circ f_2(s) = \sum_{n=1}^{\infty} \lambda_n !\ a_n b_n e^{s\lambda_n}, \text{ where}$$

$$f_1(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \text{ and } f_2(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$$

are elements of Ω Evidently, $\phi(s) = \sum_{n=1}^{\infty} 0_n e^{s\lambda_n}$, where $0_n = 0$, for all n, is the null element of Ω. The indentity element of Ω is given by

$$e(s) = \sum_{n=1}^{\infty} (1/\lambda_n!)\ e^{s\lambda_n} .$$

Clearly, $f(s) \circ e(s) = e(s) \circ f(s) = f(s)$ for any $f \in \Omega$. The negative element of $\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ is $\sum_{n=1}^{\infty} (-a_n) e^{s\lambda_n}$. In this paper our aim is to study a few algebraic structures of the space Ω. We shall also study some growth properties of the entire functions belonging to Ω.

Theorem—1 : Ω is a normed complex linear space.

Proof : We observe that Ω is an additive abelian group. Now we shall show that Ω is a linear space over the field C of complex numbers.

Easily, we can verify the following :

i) $1f_1(s)=f_1(s)$

ii) $(\alpha+\beta)\, f_1(s)=\alpha f_1(s)+\beta f_1(s)$

iii) $(\alpha\beta)\, f_1(s)=\alpha(\beta f_1(s))$

iv) $\alpha(f_1(s)+f_2(s))=\alpha f_1(s)+\alpha f_2(s)$

where $\alpha,\beta \in C$ and $f_1, f_2 \in \Omega$. Hence Ω is a complex linear space.

Next, we define the norm of $f_1 \in \Omega$ by $\| f_1(s) \| = \sup_{n \geqslant 1}\{\lambda_n! \mid a_n \mid\}$.

Since $\lambda_n ! \mid a_n \mid$ is bounded, $\sup\limits_{n \geqslant 1} \lambda_n ! \mid a_n \mid$ exists. Now,

i) $\| f_1(s) \| \geqslant 0$ and $\| f_1(s) \| = 0$ iff $f_1(s) = 0$.

$$\text{(ii)} \quad \| f_1(s) + f_2(s) \| = \left\| \sum_{n=1}^{\infty} (a_n + b_n) e^{s\lambda_n} \right\|$$

$$= \sup_{n \geqslant 1} \{ \lambda_n ! \mid a_n + b_n \mid \}$$

$$\leqslant \sup_{n \geqslant 1} \lambda_n ! \mid a_n \mid + \sup_{n \geqslant 1} \lambda_n ! \mid b_n \mid$$

$$= \| f_1(s) \| + \| f_2(s) \| .$$

$$\text{(iii)} \quad \| \alpha f_1(s) \| = \left\| \sum_{n=1}^{\infty} \alpha a_n e^{s\lambda_n} \right\| = \sup_{n \geqslant 1} \{ \lambda_n ! \mid \alpha a_n \mid \}$$

$$= \mid \alpha \mid \sup_{n \geqslant 1} \{ \lambda_n ! \mid a_n \mid \} = \mid \alpha \mid \| f_1(s) \|$$

Hence Ω is a normed complex linear space.

Theerem—2 : Ω is a complex Banach space.

Proof . Let the sequence $\{ f_p \}$, where $f_p(s) = \sum_{n=1}^{\infty} a_{p,n} e^{s\lambda_n} \in \Omega$

for $p = 1, 2, \ldots\ldots$, be a Cauchy sequence. So for every $\xi > 0$ there exists a positive integer N such that

$$d(f_p, f_q) = \| f_p - f_q \| < \epsilon \text{ for } p,q \geqslant N. \text{ Hence}$$

$$\sup \{ \lambda_n ! \mid a_{p,n} - a_{q,n} \mid \} < \epsilon \text{ for } p,q \geqslant N. \text{ This implies}$$

that

(1.4) $\lambda_n ! \mid a_{p,n} - a_{q,n} \mid < \epsilon$ for $p,q \geqslant N$ and for every n. We now fix n and consider the sequence $\{a_{1,n}\ s_{2,n} \ldots\ldots\}$, Due to (1.4) this sequence converges to α_n (say). Hence

$$\lambda_n ! \mid a_{p,n} - a_n \mid < \epsilon \text{ for } p \geqslant N \text{ and for all n.}$$

We now form the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \alpha_n e^{s\lambda_n}$. To complete the proof we have to show that $f \in \Omega$. Now.

$$\begin{aligned}\lambda_n!\ |\alpha_n!| &= \lambda_n!\ |\alpha_n - a_{p,n} + a_{p,n}| \\ &\leqslant \lambda_n!\ |\alpha_n - a_{p,n}| + \lambda_n!\ |a_{p,n}| \\ &< \epsilon + k_1 \leqslant K \text{ (say)}\end{aligned}$$

Therefore, $f \in \Omega$ and this completes the proof.

Theorem—3 : Ω is a commutative Banach algebra with identity.

Proof : It follows easily that Ω is a commutative algebra over C. Now, let $f_1, f_2, \in \Omega$. Then,

$$\| f_1 o f_2 \| = \left\| \sum_{n=1}^{\infty} \lambda_n!\ a_n b_n e^{s\lambda^n} \right\| = \sup_{n \geqslant 1} \{[\lambda_n!]^2\ |a_n b_n|\}$$

$$\leqslant \sup_{n \geqslant 1} \{\lambda_n!\ |a_n|\} \sup_{n \geqslant 1} \{\lambda_n!\ |b_n|\} = \| f_1(s) \|\ \| f_2(s) \|.$$

Again for the identity element e(s), we have,

$$\| e(s) \| = \sup_{n \geqslant 1} \{\lambda_n! \frac{1}{\lambda_n!}\} = 1.$$

Hence the theorem.

Theorem—4 : Ω is a commutativ B* — algebra with identity.

Proof . Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ and $f_2(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$ be two elements of Ω,

We define a mapping $\Psi : \Omega \to \Omega$ such that $\Psi(f_1) = f_1^*$, where

$f_1^*(s) = \sum_{n=1}^{\infty} \bar{a}_n e^{s\lambda_n}$, $\bar{a}_n$ denotes the conjugate of a_n. Evidently $f_1^* \in \Omega$.

We can verify easily the following :

i) $\psi(f_1+f_2)=(f_1+f_2)^*=f_1^*+f_2^*$

ii) $\psi(\alpha f_1)=(\alpha f_1)^* = \bar{\alpha} f_1^*, \quad \alpha \in C$

(iii) $\Psi(f_1 \circ f_2) = (f_1 \circ f_2)^* = \sum_{n=1}^{\infty} \lambda_n ! \; \overline{a_n b_n} \, e^{s\lambda_n}$ and

$$\Psi(f_2) \circ \Psi(f_1) = f_2^* \circ f_1^* = \sum_{n=1}^{\infty} \lambda_n ! \; \overline{b_n a_n} \, e^{s\lambda_n}$$

so that $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$

(iv) $(f_1^*)^* = f_1$.

Thus, Ψ is an involution in Ω. Hence Ω is a commutative Banach* - algebra with identity. Again,

$\| f_1^* \circ f_1 \| = \| f_1 \|^2$, $f_1 \in \Omega$. Thus, Ω is a commutative B* - algebra with identity.

Theorem—5 : If $f_1 \in \Omega$, then $\mathrm{Sp}(f_1)$ = closure of the set

$$\{\lambda_n ! \; a_n \;:\; n=1, 2, \ldots\ldots\}$$

where $\mathrm{Sp}(f_1)$ = Spectrum of f_1.

To find the spectrum of an element in Ω we need the following lemma.

Lemma : Let $f_1 \in \Omega$. Then f_1 is invertible iff $a_n \neq 0$ for all $n \geqslant 1$ and there exists a constant M such that

$$1/(\lambda_n ! \; | a_n |) \leqslant M \text{ for all } n \geqslant 1.$$

Proof of the lemma : If f_1 is invertible then there exists an element

$f_2(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n} \in \Omega$ such that

$$(f_1 \circ f_2)(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n !} e^{s\lambda_n} = e(s). \quad \text{Hence.}$$

$$\lambda_n ! \; a_n b_n = \frac{1}{\lambda_n !} \text{ for all } n \geqslant 1.$$

It follows that $a_n \neq 0$ for all n. Thus,

$$\lambda_n ! \; b_n = \frac{1}{\lambda_n ! \, a_n}.$$

Since $f_2 \in \Omega$ it follows that $1/(\lambda_n !\ a_n) \leqslant M$ for all $n \geqslant 1$.

Conversely, let us consider $f_2(s) = \sum_{n=1}^{\infty} 1/((\lambda_n!)^2 a_n)\, e^{s\lambda_n}$. Then $f_2 \in \Omega$

$$\text{and} \quad (f_1 \circ f_2)(s) = \sum_{n=1}^{\infty} \lambda_n !\ a_n \frac{1}{(\lambda_n !)^2 a_n} e^{s\lambda_n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n !} e^{s_n \lambda_n} = e(s) \text{ which}$$

proves that f_1 is invertible.

Proof of the theorem : We know $\alpha \in Sp(f_1)$ iff $f_1 - \sigma e$ is not invertible. By lemma it follows that this is possible iff the sequence

$$\left\{ \frac{1}{\lambda_n !\, \left| a_n - \frac{\alpha}{\lambda_n !} \right|} \right\} \text{ is unbounded. This happens}$$

iff there exists sub sequences $\{M_p\}$ such that $|\ \lambda_n !\ a_n - \alpha\ | \to 0$ for $n = M_p$, $p = 1, 2, \ldots\ldots\ldots$, that is, iff $\alpha \in$ closure of $\{\lambda_n !\ a_n\}$, $n = 1, 2, \ldots\ldots\ldots$ Hence the theorem.

Theorem—6 : Ω itself is the set of its topological divisors of zero.

Proof : Since Ω is commutative, any topological divisor of zero in Ω is a two sided topological divisor of zero in Ω. In Ω we consider a sequence $\{w_n\}$ where $w_n = 1/\lambda_n!\ e^{s\lambda_n}$. Obviously, $\|\ w_n\ \| = 1$ for all n.

Now for any arbitrary element $f_1(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \in \Omega$, we have $(f_1 \circ w_n)(s) = a_n e^{s\lambda_n}$.

Now $f_1 \in \Omega$ implies $\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ is convergent for all s and hence

$$\lim_{n \to \infty} a_n e^{s\lambda_n} = 0.$$

Thus f_1 is a topological divisor of zero in Ω. Since f_1 is arbitrary, any element in Ω is a two sided topological divisor of zero which proves the theorem.

2. In this section we shall point out some of the growth properties of elements in Ω.

Let ρ and δ be the linear order and lower linear order of the entire function $f \epsilon \Omega$ where f is given by (1.1). Then it is known [1] that

$$\text{(2.1)} \qquad \limsup_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \rho \ , \quad (0 \leqslant \rho \leqslant \infty)$$

Again if $\log \lambda_{n+1}$ $\log \lambda_n$ and $\dfrac{\log |a_n / a_{n+1}|}{\lambda_{n+1} - \lambda_n}$ is a non-decreasing function of n for $n \geqslant n_0$ then [2]

$$\text{(2.2)} \qquad \liminf_{n\to\infty} \frac{\lambda_n \log \lambda_n}{\log |a_n|^{-1}} = \delta \ , \quad (0 \leqslant \delta \leqslant \infty).$$

We say that the entire function $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ is of linearly regular growth if $\delta = \rho$.

Throughout this paper we shall assume that the linear orders of the entire Dirichlet series are finite and non-zero, and the functions satisfy (1.2).

Theorem—7 : Every element $f \epsilon \Omega$ is of linear order $\rho \leqslant 1$.

Proof : Let $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ be an entire Dirichlet series with integral exponents such that $\lambda_n ! \, |a_n| < M$ for all n. By (2.1) if ρ is the linear order of $f \epsilon \Omega$ then

$$\frac{1}{\rho} = \liminf_{n\to\infty} \frac{\log (1/|a_n|)}{\lambda_n \log \lambda_n} \geqslant \liminf_{n\to\infty} \frac{-\log (M/\lambda_n !)}{\lambda_n \log \lambda_n}$$

$$\geqslant \liminf_{n\to\infty} \frac{\log (1/M)}{\lambda_n \log \lambda_n} + \liminf_{n\to\infty} \frac{\log \lambda_n !}{\lambda_n \log \lambda_n} = 1$$

[using Stirling's formula]

Theorem—8 : Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ and $f_2(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$ be two elements of Ω of linear orders ρ_1 and ρ_2 respectively. Then (i) the linear order ρ of $f_1(s) \circ f_2(s)$ satisfies

$$\frac{1}{\rho} \geqslant \frac{1}{\rho_1} + \frac{1}{\rho_2} - 1$$

and (ii) when $f_1(s)$ and $f_2(s)$ are of linearly regular growth and $| a_n / a_{n+1} |$, $| b_n / b_{n+1} |$ be two non-decreasing functions for $n \geqslant n_0$ and

$$\log \lambda_{n+1} \quad \log \lambda_n, \text{ then } \frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} - 1.$$

Proof : We have, $f_1(s) \text{ o } f_2(s) = \sum_{n=1}^{\infty} \lambda_n ! \; a_n b_n \; e^{s\lambda_n}$. Now,

$$\frac{1}{\rho} = \liminf_{n\to\infty} \frac{\log | 1/\lambda_n ! \, a_n b_n |}{\lambda_n \log \lambda_n}$$

$$\geqslant \liminf_{n\to\infty} \frac{\log (1/\lambda_n !)}{\lambda_n \log \lambda_n} + \liminf_{n\to\infty} \frac{\log (1/ | a_n |)}{\lambda_n \log \lambda_n}$$

$$+ \liminf_{n\to\infty} \frac{\log (1/ | b_n |)}{\lambda_n \log \lambda_n}.$$

Hence (2.3) $\frac{1}{\rho} \geqslant -1 + \frac{1}{\rho_1} + \frac{1}{\rho_2}$. This proves (i). Again,

$$\frac{1}{\rho} = \liminf_{n\to\infty} \frac{\log (1/ | \lambda_n ! \, a_n b_n |)}{\lambda_n \log \lambda_n} \leqslant \limsup_{n\to\infty} \frac{\log (1/ | \lambda_n ! \; a_n b_n |)}{\lambda_n \log \lambda_n}$$

$$\leqslant \limsup_{n\to\infty} \frac{\log (1/\lambda_n !)}{\lambda_n \log \lambda_n} + \limsup \frac{\log (1/ | a_n |)}{\lambda_n \log \lambda_n}$$

$$+ \limsup_{n\to\infty} \frac{\log (1/ | b_n |)}{\lambda_n \log \lambda_n} \; ..$$

(2.4) $\frac{1}{\rho} \leqslant -1 + \frac{1}{\delta_1} + \frac{1}{\delta_2}$, [using (2.2)]

where δ_1 and δ_2 are lower linear orders of f_1 and f_2 respectively. Therefore, when $f_1(s)$ and $f_2(s)$ are of linearly regular growth then from (2 3) and (2.4), we have,

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} - 1 .$$

This proves (ii).

Theorem—9 : If $f_1(s)$ and $f_2(s)$ be two elements of Ω then $f_1^k(s) \text{ o } f_2^k(s)$,

$k=o, 1, 2, \ldots\ldots$ is also an element of Ω and its linear order is the same as that of $f_1(s) \circ f_2(s)$, where $f_1^k(s)$ denotes the k-th derivative of $f_1(s)$.

Proof : Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ and $f_2(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$ be elements of Ω.

Then, $f_1^k(s) \circ f_2^k(s) = \sum_{n=1}^{\infty} \lambda_n ! \, \lambda_n^{2k} a_n b_n e^{s\lambda_n} = \sum_{n=1}^{\infty} C_n e^{s\lambda_n}$, (say),

where $C_n = \lambda_n ! \ \lambda_n^{2k} a_n b_n$. Now it is easy to verify that $\lambda_n ! \mid C_n \mid$ is bounded. This proves that $f_1^k \circ f_2^k \in \Omega$.

The linear order ρ_k, of $f_1^k(s) \circ f_2^k(s)$ is

$$\frac{1}{\rho_k} = \liminf_{n\to\infty} \frac{\log (1/\mid C_n \mid)}{\lambda_n \log \lambda_n} = \liminf_{n\to\infty} \frac{\log (1/\mid \lambda_n ! \, \lambda_n^{2k} a_n b_n \mid)}{\lambda_n \log \lambda_n}$$

$$= \liminf_{n\to\infty} \frac{\log (1/\lambda ! \mid a_n b_n \mid)}{\lambda_n \log \lambda_n} = \frac{1}{\rho}$$

where ρ is the linear order of $f_1(s) \circ f_2(s)$.

Theorem—10 : Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ and $f_2(s) = \sum_{n=1}^{\infty} b_n e^{s\lambda_n}$ be two elements of Ω. Then $f_1(s) * f_2(s) = \sum_{n=1}^{\infty} a_n b_n e^{s\lambda_n}$ is also an element of Ω and the linear order ρ^* of $f_1(s) * f_2(s)$ will be related to the linear order ρ of $f_1(s) \circ f_2(s)$ by the relation

$$\frac{1}{\rho} = \frac{1}{\rho^*} - 1.$$

Proof : Since $f_1, f_2 \in \Omega$, $\lambda_n ! \mid a_n \mid < M_1$ and $\lambda_n ! \mid b_n \mid < M_2$ for all n. It follows that $\lambda_n ! \mid a_n b_n \mid$ is bounded. So, $f_1(s) * f_2(s)$ is an element of Ω.

If $f_1(s) * f_2(s)$ is either zero or a Dirichlet polynomial then $\rho^* = o$. In that case the linear order ρ is also zero

We now assume that $f_1(s) \circ f_2(s)$ is different from Dirichlet polomomial. Then.

$$\begin{aligned}\frac{1}{\rho} &= \liminf_{n\to\infty} \frac{\log (1/\lambda_n! \mid a_n b_n \mid)}{\lambda_n \log \lambda_n} \\ &\leqslant \limsup_{n\to\infty} \frac{\log (1/\lambda_n!)}{\lambda_n \log \lambda_n} + \liminf_{n\to\infty} \frac{\log (1/ \mid a_n b_n \mid)}{\lambda_n \log \lambda_n} \\ &= -1 + 1/\rho^*\end{aligned}$$

Again, $\frac{1}{\rho} = \liminf_{n\to\infty} \frac{\log (1/\lambda_n! \mid a_n b_n)}{\lambda_n \log \lambda_n}$

$$\begin{aligned}&\geqslant \liminf_{n\to\infty} \frac{\log (1/\lambda_n!)}{\lambda_n \log \lambda_n} + \liminf_{n\to\infty} \frac{\log (1/ \mid a_n b_n \mid)}{\lambda_n \log \lambda_n}. \\ &= -1 + \frac{1}{\rho^*}\end{aligned}$$

Therefore, $\frac{1}{\rho} = \frac{1}{\rho^*} - 1.$

Finally, I take the opportunity to express my thanks to Dr. B. C. Chakraborty, Reader of the Department of Pure Mathematice, C. U. for his valuable suggestions and guidance in the preparation of this paper.

References :

1. Mandelbrojit, S. : Dirichlet series. The Rice Institute Pamphlet, 31 No. 4 (1944) pp. 159-272.

2. Rahman, Q I. : On the lower order of entire functions defined by Dirichlet series Quart. J. Math. (OX), 7 (1956) pp. 96-99.

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A NOTE ON PSEUDO CONFORMALLY SYMMETRIC SPACES

U. C. DE and B. BARUA

1. Introduction. In a recent paper [3] the first author and H. A. Biswas introduced and studied a type of non-flat Riemannian space (M^n,g) $(n>3)$ whose conformal curvature tensor C^h_{ijk} defined by

$$(1.1)\quad C^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(g_{ij}R^h_k - g_{ik}R^h_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ij}\,\delta^h_k - g_{ik}\,\delta^h_j)$$

satisfies the condition

$$(1.2)\quad C^h_{ijk,l} = 2\,\lambda_l C^h_{ijk} + \lambda^h C_{lijk} + \lambda_i\, C^h_{ljk} + \lambda_j\, C^h_{ilk} + \lambda_k\, C^h_{ijl}$$

where R^h_{ijk} is the curvature tensor ot type (1,3), R^h_k is the (1,1) Ricci tensor, R is the scalar curvature, λ_l is a non-zero covariant vector and comma ',' denotes covariant differentiation with respect to the metric tensor g_{ij}. Such a space is called a Pseudo conformally symmetric space and λ_l is called its associated vactor. An n-dimensional space of this kind has been denoted by $(PCS)_n$. If λ_l becomes zero in (1.2), then the space reduces to a conformally symmetric space [2].

It is to be noted that dimension of M^n has been taken >3, because it is known that $C^h_{ijk} = 0$, when $n = 3$.

In the present paper some results of a $(PCS)_n$ $(n>3)$ are established. At first it is shown that the associated vector field of a $(PCS)_n$ is necessarily a gradient. Next a necessary and sufficient condition for constant scalar curvature of a $(PCS)_n$ has been obtained. In the last section Ricci - recurrent $(PCS)_n$ has been studied.

2. Preliminaries. The conformal curvature tensor satisfies the well known equations :

$$(2.1)\quad C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi}$$

where $C_{hijk} = g_{ph}C^p_{ijk}$

$$(2.2)\quad C_{hijk} + C_{hjki} + C_{hkij} = 0,\quad C^r_{ijr} = C^r_{irk} = C^r_{rjk} = 0$$

(2.3) $C^{r}_{ijl,r} = \frac{n-3}{n-2}\left[(R_{ij,l}-R_{il,j}) - \frac{1}{2(n-1)}(R_{,l}g_{ij} - R_{,j}g_{il})\right]$

The above results will be used in the next sections.

3. Nature of the associated vector field λ_i of a $(PCS)_n$ $(n \geq 3)$.

Contracting (1.2) we get by using (2.2)

(3.1) $\lambda^{h}C_{lijh} + \lambda_{h}C^{h}_{ijl} = 0$

Using (2.1) we get from (3.1)

(3.2) $\lambda_{h}C^{h}_{ilj} = \lambda_{h}C^{h}_{jil}$

Again using (2.2) it follows from (3.2)

(3.3) $\lambda_{h}C^{h}_{.lij} = 0$

Now (1.2) can be written as

(3.4) $C_{hijk,l} = 2\lambda_{l}C_{hijk} + \lambda_{h}C_{lijk} + \lambda_{i}C_{hljk} + \lambda_{j}C_{hilk} + \lambda_{k}C_{hijl}$

where $C_{hijk} = g_{ph}C^{p}_{ijk}$

Transvecting (3.4) with C^{hijk} and using (3.3) we get

(3.5) $A_{l} = 4A\lambda_{l}$

where $A = C^{hijk}C_{hijk}$

Since $A \neq 0$ at least at some point, (3.5) yields $A \neq 0$ everywhere. Thus from (3.5) we have

$\lambda_{i} = \frac{1}{4}\,\delta_{i} \log A,$

We can therefore state the following theorem :

Theorem 1. The associated vector field of a $(PCS)_n$ is necessarily a gradient.

A Riemannian or Pseudo-Riemannian space (M^{n},g) is said to be of harmonic conformal curvature [1] if $n \geq 4$ and the condition

$C^{h}_{ijk,h} = 0$ holds.

Now contracting (1.2) and using (3.3) we get

(3.6) $C^{h}_{ijk,h} = 0$

Thus we have the following theorem :

Theorem 2. Every Pseudo conformally symmetric space $(PCS)_n$ $(n \geq 4)$ is of harmonic conformal curvature.

4. Nature of the scalar curvature of a $(PCS)_n$.

The scalar curvature of a $(PCS)_n$ is not necessarily a constant. We therefore look for a necessary and sufficient condition for a $(PCS)_n$ to be of constant scalar curvature. For this we require the following definition:

The Ricci tensor is said to be of Codazzi type [4] if the Ricci tensor satisfies the condition

(4.1) $R_{ij,l} = R_{il,j}$

From Theorem 2 and (2.3) it follows that

(4.2) $R_{ij,l} - R_{il,j} - \frac{1}{2(n-1)}(R_{,l}\, g_{ij} - R_{,j} g_{il}) = 0$

If R is constant, then we have from (4.2) that the Ricci tensor is a Codazzi tensor. Conversely, if the Ricci tensor is a Codazzi tensor, then from (4.2) we get

$R = \text{constant}$

We can therefore state the following theorem ;

Theorem 3. The scalar curvature of a $(PCS)_n$ is constant if and only if the Ricci tensor is a Codazzi tensor.

5. Ricci-recurrent $(PCS)_n$.

Let a $(PCS)_n$ be a Ricci recurrent space [5] specified by a non-zero vector μ_i, that is,

(5.1) $R_{ij,l} = \mu_l R_{ij}$

It is known that for a non-decomposable Ricci-recurrent space, $R = 0$. Hence from Theorem 2 and (2.3) it follows that for a non-decomposable Ricci-recurrent $(PCS)_n$,

$R_{ij,k} = R_{ik,j}$

Hence

$\mu_k R_{ij} = \mu_j R_{ik}$

Therefore

(5.4) $R_{ij} = S\mu_i\mu_j$ $(S \neq 0)$

where S is a scalar factor of proportionality.

From (5.1) it follows

$g^{ij}R_{ij} = S\ g^{ij}\mu_i\mu_j$

whence $g^{ij}\mu_i\mu_j = 0$

Thus we have the following theorem :

Theorem 4. If a non-decomposable $(PCS)_n$ is Ricci-recurrent then the vector of recurrence μ_i is a null vector and the rank of the matrix (R_{ij}) is one.

Now we suppose that a $(PCS)_n$ is a Ricci-recurrent space with the same associated vector field λ_i, that is,

$$(5.2)\quad R_{ji,l} = \lambda_l R_{ij}$$

Then from (4.2) and (5.2) it follows that

$$(5.3)\quad \lambda_l R_{ij} - \lambda_j R_{il} = \frac{1}{2(n-1)}\left[\lambda_l R g_{ij} - \lambda_j R g_{il}\right]$$

Transvecting (5.3) with g^{ij} we get

$$\lambda^i R_{il} = \frac{1}{2}\lambda_l R$$

whence

$$(5.4)\quad \frac{R_{il}\lambda^i\lambda^l}{g_{ij}\lambda^i\lambda^l} = \frac{R}{2}$$

Hence we can state the following theorem :

Theorem 5. In a Ricci-recurrent $(PCS)_n$ the Ricci curvature in the direction of the associated vector field λ^i is $\frac{R}{2}$.

References

[1] Besse A. L : Einstein Manifolds, Springer Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo 1987.

[2] Chaki M. C. and Gupta B. : On conformally symmetric spaces, Indian J. Math. 5, 1963, 113-122.

[3] DE U. C. and Biswas H. A. : On Pseudo conformally symmetric manifolds, Bull. Cal, Math. Society 85, (1993) 479-486.

[4] Ferus D. : A remark on Codazzi tensors in constant curvature spaces, Lecture notes in Mathematics, 838, Global Differential Geometry and global analysis, Springer Verlag, New York, 1981.

[5] Patterson E. M. : Some theorems on Ricci-recurrent spaces, J. London Math, Soc. Vol. 27, 1952, 287-295.

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ENTIRE FUNCTIONS OF BOUNDED INDEX IN SEVERAL COMPLEX VARIABLES

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[**Abstract** : In this paper we first introduce the concept of bounded index (b.i.) for an entire function in several complex variables. We then study a few characteristic properties of such functions. We have also shown that the sum of two entire functions of bounded index in C^n need not be of bounded index. Finally we have obtained a necessary condition for an entire function in C^n to be of bounded index and suggested a problem for the sufficient condition.]

1. **Introduction** : The concept for an entire function of bounded index in C, the complex plane, was first introduced by B. Lepson [3] in 1968. Following his preliminary investigations many mathematicians like Fred Gross, S. M. Shah, G. H. Fricke, B. S. Lee etc. obtained a few more properties and characterization of such functions. It turned out that solutions of a large class of differential equations in C are entire functions of b. i. Unfortunately, almost no work has ben done on the corresponding theory in several complex variables. Till date, to our knowledge, only in one paper J. G. Krishna and S. M. Shah[2] generalized this concept for analytic functions in several complex variables with respect to an arbitrary given point α in C^n, complex n-space. We now introduce the concept of b.i. for entire functions in several variables without reference to any given point. In fact, our definition coincides with that of J. G. Krishna and S. M. Shah if the analytic function is, in particular, an entire function and the arbitrary given point $\alpha = (1, \ldots\ldots, 1)$.

2. **Notations** : We denote complex and real n-space by C^n and R^n respectively and the set of non-negative integers by I.

$I_+^n = \{ x : x \varepsilon I^n, x_i > 0, i = 1, \ldots\ldots, n\}$. We indicate the points $(z_1, \ldots, z_n)$, $(w_1, \ldots, w_n)$ etc. of C^n by their corresponding unsuffixed symbols z, w etc. For z, w ε C^n and $\alpha \varepsilon C$ we define $\alpha z = (\alpha z_1, \ldots\ldots, \alpha z_n)$, $z+w = (z_1 + w_1, \ldots\ldots, z_n + w_n)$, $z.w = z_1 w_1 + \ldots + z_n w_n$, $|z| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2}$. The positive hyperoctant R_+^n in R^n will be $R_+^n = \{ x : x \varepsilon R^n, x_i \geq 0, i = 1, \ldots, n\}$. For $t \varepsilon R_+^n$ we set $\|t\| = t_1 + \ldots + t_n$ and for $m \varepsilon I^n$, $m! = m_1! \ldots m_n!$. $z^m = z_1^{m_1}, \ldots z_n^{m_n}$ for $z \varepsilon C^n$, $m \varepsilon I^n$ ($z_i^0 = 1$ even if $z_i = 0$). For any $k \varepsilon R$, $\bar{k} = (k, \ldots, k)$. In particular, $\bar{0}$ and $\bar{\infty}$ will respectively denote the n-tuples $(0, \ldots, 0)$ and $(\infty, \ldots, \infty)$. Also for $x, y \varepsilon R^n$ we say that

(i) $x \leq y$ iff $x_i \leq y_i$; (ii) $x < y$ iff $x \leq y$ but $x \neq y$;

(iii) $x << y$ iff $x_i < y_i$, $i = 1, \dots, n$.

For an entire function f with domain C^n, $f^{(k)}$ will denote the partial derivative

$$\frac{\delta^{\|k\|} f}{\delta z_1^{k_1} \dots \delta z_n^{K_n}} \text{ where } k \,\varepsilon\, I^n \text{ and } f^{(O)} = f.$$

3. An entire function f in C^n has a taylor expansion about any point $w \,\varepsilon\, C^n$ of the form :

$f(z) = \sum_{m=0}^{\infty} a_m (z-w)^m$, where $a_m = \frac{f^{(m)}(w)}{m!}$. Since the power series of f about w is absolutely convergent everywhere in C^n, the terms $\frac{|f^{(m)}(w)|}{m!}$ must tend to zero. Consequently, for each $w \,\varepsilon\, C^n$ there is an n-tuple $(m_1, \dots, m_n) = m \,\varepsilon\, I^n$, $m = m(w)$, such that $\frac{|f^{(m)}(w)|}{m!}$ is a maximal coefficient. Thus,

Definition 1. For each point $w \,\varepsilon\, C^n$ there exists an n-tuple $m(w) \,\varepsilon\, I^n$ such that $\max_{p \le m} \left[\frac{|f^{(p)}(w)|}{p!}\right] \ge \left[\frac{|f^{(q)}(w)|}{q!}\right]$ for all $q \,\varepsilon\, I^n$.

There may be several tuples m for which the above inequality holds. A minimal element of such tuples is called an index of f at the point w

Definition 2. An entire function f in C^n is said to be of bounded index iff there exists an $m \,\varepsilon\, I^n$ such that for all $z \,\varepsilon\, C^n$,

$$(3.1) \quad \max_{p \le m} \left[\frac{|f^{(p)}(z)|}{p!}\right] \ge \frac{|f^{(q)}(z)|}{q!} \text{ for all } q \,\varepsilon\, I^n.$$

There may have several tuples satisfying (3.1). A minimal element of such tuples is called an index of f and is denoted by I(f). An entire function which is not of b.i. is said to be of unbounded index.

Examples : The polynomials in $z_1, \dots, z_n$; the functions $\exp(\alpha(z_1 + \dots + z_n))$, $\alpha \,\varepsilon\, C$; $\sin(z_1 \pm \dots \pm z_n)$; $\cos(z_1 \pm \dots \pm z_n)$ etc are of b.i. whereas the function $\exp(z_1^2 + \dots + z_n^2)$ is not of b.i. For the function $f(z) = \exp(\alpha(z_1 + \dots + z_n))$, $\alpha \,\varepsilon\, C$, we observe that $I(f) = ([|\alpha|], \dots, [|\alpha|])$, where [ ] denotes the step function while an index of a polynomial P(z) with degree $\|m\|$ is such that $I(P) \le m$.

Theorem 1. Let f be analytic in $D = D_1 \times \dots \times D_n$ where $D_i = \{z_i : |z_i - z_i^{(O)}| \le 1+\delta_i, \ \delta_i > 0\}$, $i = 1, \dots, n$. Then the sequence $\left\{\frac{f^{(m)}(z)}{m!}\right\}$ converges to zero uniformly in the closed region $K = K_1 \times \dots \times K_n$, where $K_i = \{z_i : \left|z_i - z_i^{(o)}\right| \le \delta_i\}$.

Proof : Since the point $\bar{1} + z^{(O)}$ lies in D, the sequence of coefficients $\left\{\frac{f^{(m)}(z)}{m\,!}\right\}$ of the Taylor series expansion of f converges pointwise to zero in D. Let M = max $|f(z)|$ on the distinguised boundary of D. By Cauchy's integral formula, for any z in K,

$$\left|\frac{f^{(m)}(z)}{m\,!}\right| = \frac{1}{(2\pi)^{n}} \left|\int_{C} \frac{f(w)\, dw_1 \ldots dw_n}{(w-z^{(0)})^{m+\bar{1}}}\right| \text{ where } C = C_1 x \ldots . x C_n,$$

$$C_i = \{ z_i : |z_i - z_i^{(O)}| = 1 + \delta_i \}$$

$$\leq \frac{1}{(2\pi)^n} \cdot \frac{(2\pi)^n \prod_{i=1}^{n} (1+\delta_i) M}{\prod_{i=1}^{n} (1+\delta_i)^{m_i+1}} < M.$$

Hence by Vitali's theorem the sequence. $\left\{\frac{f^{(m)}(z)}{m\,!}\right\}$ converges uniformly in any compact subset of int K.

Hence if we replace δ_i's in K_i by slightly larger values, the conclusion follows.

Corollary : Let f be an entire function in C^n and $E = E_1$ x...x E_n be any bounded set. Then $\left\{\frac{f^{(m)}(z)}{m\,!}\right\}$ converges to zero uniformly in E.

Theorem 2. For $\alpha \in C^n$, $|\alpha_i| > 1$, the function $f(z) = Qe^{\alpha.z}$ is of bounded index whenever Q is a polynomial.

Proof : For the sake of simplicity we shall prove the result for two variables only. The generalization to n variables is immediate.

Suppose that Q is a polynomial of degree m_1+m_2. We may assume without any loss of generality that the leading coefficient of Q is 1. Now,

$$\frac{f^{(k_1+k_2)}(z)}{e^{\alpha.z}} = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} {}^{k_1}C_i \; {}^{k_2}C_j \; Q^{(i+j)}(z) \alpha_1^{k_1-i} \alpha_2^{k_2-j}$$

for $1 \leq i \leq k_1$, $1 \leq j \leq k_2$ and ${}^{k_1}C_i = {}^{k_2}C_j = 0$ for $i > k_1, j > k_2$.

We observe that ${}^{k_1}C_i < k_1{}^{m_1}$, ${}^{k_2}C_j < k_2{}^{m_2}$. Now using the inequality of maximum modulus of a polynomial we have,

$|Q^{(i+j)}(z)| \leq |a_{m_1,m_2}| \, r_1^{m_1} r_2^{m_2}(1+\varepsilon) = 3/2 \, r_1^{m_1} r_2^{m_2}$, (taking $\varepsilon = {}^1/_2$) for sufficiently large $\|r\|$ and for all i, j ε I, where $r_1 = |z_1|$, $r_2 = |z_2|$.

If possible, suppose that f is of unbounded index. Then there exists a double sequence $\{k_1, k_2\}$ of positive integers and a corresponding sequence of points $\{(z_{1,k_1}, z_{2,k_2})\}$ such that $k_1+k_2 \to \infty$ and $z_{1,k_1} + .z_{2,k_2} \to \infty$ which will satisfy the inequality $|f(z_{1,k_1}, z_{2,k_2})| < \frac{1}{k_1!\,k_2!} |f^{(k_1+k_2)}(z_{1,k_1}, z_{2,k_2})|$.

This implies that

$$|Q(z_{1,k_1}, z_{2,k_2})| < \frac{1}{k_1!\,k_2!} \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} {}^{k_1}C_i \; {}^{k_2}C_j \; Q^{(i+j)} \; (z_{1,k_1}, z_{2,k_2}) \alpha_1^{k_1-i} \; \alpha_2^{k_2-j}$$

That is,

$$|Q(z_{1,k_1}, z_{2,k_2})| < \frac{3}{2k_1!k_2!} \sum_{i=o}^{m_1} \sum_{j=o}^{m_2} (k_1^{m_1} \; k_2^{m_2} |\alpha_1|^{k_1-i} |\alpha_2|^{k_2-j} \; r_{1,k_1}^{m_1} r_{2,k_2}^{m_2})$$

That is

$$(3.2) \quad |Q(z_{1,k_1}, z_{2,k_2})| < \frac{3}{2k_1!\,k_2!}\left[k_1^{m_1} k_2^{m_2} r_{1,k_1}^{m_1}, r_{2,k_2}^{m_2}, |\alpha_1|^{k_1} |\alpha_2|^{k_2} (m_1+1)(m_2+1)\right]$$

for sufficiently large $r_{1,k_1}+r_{2,k_2}$ where $|z_1,k_1|=r_1,k_1, |z_2,k_2|=r_2,k_2$. Again applying the inequality of minimum modulus of a polynomial we get,

(3.3) $|Q(z_{1,k_1}, z_{2,k_2})| > \frac{1}{2} r_{1,k_1}^{m_1} r_{2,k_2}^{m_2}$ for sufficiently large $r_{1,k_1}+r_{2,k_2}$. Combining (3.2) and (3.3) we get,

$$\frac{1}{2} r_{1,k_1}^{m_1} r_{2,k_2}^{m_2} < |Q(z_1,k_1,z_2,k_2)|$$

$$< \frac{3}{2k_1!k_2!}[(m_1+1)(m_2+1) k_1^{m_1} k_2^{m_2} r_{1,k_1}^{m_1} r_{2,k_2}^{m_2} |\alpha_1|^{k_1} |\alpha_2|^{k_2}]$$

Hence,

$$(3.4) \quad 1 < \frac{3}{k_1!k_2!}(m_1+1)(m_2+1) k_1^{m_1} k_2^{m_2} |\alpha_1|^{k_1} |\alpha_2|^{k_2}$$

for sufficiently large k_1+k_2. But (3.4) is impossible since the right hand side of (3.4) tends to zero as $k_1+k_2 \to \infty$ and the theorem follows.

4. Let g be an entire function and $M(r;g) = \max_{|z_i|=r_i} |g(z)|$. For any $k \in I^n$, we define the non-negative function $\Omega(z) = \Omega_k(z,g) = \max_{0 \le s \le k} \frac{|g^{(s)}(z)|}{s!}$

In particular, if k is an index of g, then

(4.1) $\frac{|g^{(p)}(z)|}{p!} \leq \Omega(z)$ for all z and all $p \varepsilon I^n$.

We can easily find functions g with an index k such that (4.1) may be replaced by

(4.2) $\frac{|g^{(p)}(z)|}{p!} \leq \alpha\Omega(z)$, $(p_i = k_i+1, k_i+2, \ldots., 0<\alpha<1)$, α being a constant.

Theorem 3. Let g be an entire function of bounded index with an index k and satisfying the condition (4.2). Let f be any entire function such that

(4.3) $f(\bar{O}) \neq O$ and

(4.4) $M(2r;f) \leq \Omega(z)$ for all z where $r_i = |z_i|$

Then the function

(4.5) $h(z) = g(z) + bf(z)$ is of b.i with an index $\leq k$, provided the constant $b(\neq O)$ is chosen small enough.

Proof : Cauchy's inequality gives

(4.6) $\frac{|f^{(p)}(z)|}{p!} r^p \leq M(2r\,;f)$ for all z and $p \varepsilon I^n$. Hence,

(4.7) $\frac{|f^{(p)}(z)|}{p!} \leq M(\bar{2}; f)$ for $|z_i| \leq 1$.

Since $f(\bar{O}) \neq O$, $\beta = M(\bar{2}; f)/|f(\bar{0})| \geq 1$. From (4.6) and (4.7) we have, for $|z_i| = r_i$,

(4.8) $\frac{|f^{(p)}(z)|}{p!} \leq \beta M(2r; f)$ for all $p \varepsilon I^n$ and for all z.

Differentiating (4.5) p_i $(\geq k_i + 1)$ times and then using (4.2), (4.4) and (4.8) we get,

(4.9) $\frac{|h^{(p)}(z)|}{p!} \leq (\alpha + |b|\beta)\,\Omega(z)$, $p_i \geq k_i + 1$.

If $s \leq k$, (4.8) is still valid with p replaced by s, but (4.2) no longer holds. Then, differentiation of (4.5) s times yields

(4.10) $\frac{|h^{(s)}(z)|}{s!} \geq \frac{|g^{(s)}(z)|}{s!} - |b|\beta M(2r;f)$, $(\bar{0} \leq s \leq k)$.

In view of the definition of $\Omega(z)$ and (4.4) we get,

(4.11) $\max\limits_{\bar{0} \leq s \leq k} \frac{|h^{(s)}(z)|}{s!} \geq (1-|b|\beta)\,\Omega(z)$.

If $|b|\beta < 1$ we can deduce from (4.9) and (4.11) that

$$(4.12)\quad \frac{|h^{(p)}(z)|}{p!} \le \frac{\alpha+|b|\beta}{1-|b|\beta} \max_{0\le s\le k} \frac{|h^{(s)}(z)|}{s!},\ (p_i \ge k_i+1).$$

As $b \to 0$, $\frac{\alpha+|b|\beta}{1-|b|\beta} \to \alpha < 1$. This shows that h is of b.i with an index $\le k$ if b is small enough.

Theorem 4. The sum of two entire functions of bounded index need not be of bounded index.

Proof :- If for all values of $q \in I_+^n$, $\lim_{z\to\infty} \frac{\Omega(z)}{|z^q|} = \infty$ and if $\Omega(z)$ never vanishes we can easily find an entire function f of unbounded index satisfying the conditions (4.3) and (4.4). Let f be such a function. Then, in view of theorem 3, we may choose a constant b, small enough, such that the function $bf(z) = h(z) - g(z)$. Here $bf(z)$ is of unbounded index whereas $h(z)$ and $-g(z)$ are of b.i. with an index $\le k$. This completes the proof.

5. Let $r >> \overline{O}$, $z^{(0)} \in c^n$ and $h \in I_+^n$. For an entire function f and $p \in I^n$, $p_i = 0,1,\ldots,h_i$ we define $R_{p_1}, \ldots, p_n(r,h,z^{(O)}) = R_p(r,h,z^{(O)}) = \max\left[\frac{|f^{(k)}(z)|}{k!} : |z_i - z_i^{(O)}| \le \frac{p_i r_i}{h_i}, (i=1,\ldots,n) \text{ and } k \in I^n, k_i = 0,1,\ldots\right.$

Lemma : If f is an entire function of boundd index with an index N and if r,h are such that $\frac{r_i}{h_i} \le \frac{1}{4(N_i+1)}$ (i=1,...,n), then (i) $R_t(r,h,z^{(0)}) = R_{t_1}, \ldots, t_j, \ldots, t_n(r,h,z^{(0)}) \le 2R_{t_1}, \ldots, t_j-1, \ldots t_n(r,h,z^{(0)})$, $1 \le j \le n$, for all $z^{(0)} \in C^n$ and all t with $1 \le t_i \le h_i$, and

$$\text{(ii)}\quad \max_{\substack{k\le N\\ |z_i-z_i^{(0)}|=r_i}} \left[\frac{|f^{(k)}(z)|}{k!}\right] \le 2^{\|h\|} \max_{k\le N}\left[\frac{|f^{(k)}(z^{(0)})|}{k!}\right]$$

Proof : If possible, suppose there exists a t with $1 \le t_i \le h_i$ and $z^{(0)} \in C^n$ such that $R_t(r,h,z^{(0)}) > 2R_{t_1}, \ldots, t_j-1, \ldots, t_n(r,h,z^{(0)})$.

By maximum modulus principle there exist some $z^{(t)} \in C^n$ with $|z_i^{(t)} - z_i^{(0)}| = \frac{t_i r_i}{h_i}$ and some $K^{(t)} \in I^n$ with $0 \le K_i^{(t)} \le N_i$,

(i = 1,...,n) such that $R_t(r,h,z^{(0)}) = \frac{|f^{(k^{(t)})}(z^{(t)})|}{K^{(t)}!}$. We choose a point $w^{(t)}$ such that

$w_i^{(t)} = z_i^{(t)}$, $i \neq j$ and $w_j^{(t)} = z_j^{(O)} + \frac{t_j-1}{t_j}(z_j^{(t)} - z_j^{(O)})$. Now $|w_i^{(t)} - z_i^{(O)}| = \frac{t_i r_i}{h_i}$, $i \neq j$ and $|w_j^{(t)} - z_j^{(O)}| = \frac{(t_j-1)r_j}{h_j}$. Hence, $\frac{|f^{(k^{(t)})}(w^{(t)})|}{K^{(t)}!} \leq R_{t},\dots,t_j-1,\dots,t_n\,(r,h,z^{(O)})$ and so $\frac{|f^{(k^{(t)})}(z^{(t)})|}{K^{(t)}!} - \frac{|f^{(k^{(t)})}(w^{(t)})|}{K^{(t)}!} \geq R_t(r,h,z,^{(O)}) - R_{t_1},\dots,t_j-1,\dots,t_n\,(r,h,z^{(O)})$. Again, there exists a point ξ with $\xi_i = w_i^{(t)}$, $i \neq j$ and $\xi_j = w_j^{(t)} + \lambda\,(z_j^{(t)} - w_j^{(t)})$ for some $\lambda \varepsilon\ (0.1)$ such that

$$\frac{|f^{k_1^{(t)},\dots,k_j^{(t)}+1,\dots,k_n^{(t)}}(\xi)|}{k_1^{(t)}!\dots(k_j^{(t)}+1)!\dots k_n^{(t)}!} \geq \frac{1}{k_1^{(t)}!\dots(k_j^{(t)}+1!\dots k_n^{(t)}!}\cdot\frac{|f^{(k^{(t)})}(z^{(t)})| - |f^{(k^{(t)})}(w^{(t)})|}{|z_j^{(t)} - w_j^{(t)}|}$$

$$\geq \frac{1}{k_j^{(t)}+1}\cdot\frac{R_t(r,h,z^{(0)}) - R_{t_1},\dots,t_j-1,\dots,t_n\,(r,h,z^{(0)})}{r_j/h_j}$$

$$\geq \frac{1}{N_j+1}\cdot\frac{\frac{1}{2}R_t(r,h,z^{(0)})}{\frac{1}{4}(N_j+1)} = 2R_t(r,h,z^{(0)}).$$

This is a contradiction to the choice of $R_t(r,h,z^{(0)})$ since $|\xi_j - z_j^{(0)}| < \frac{t_j r_j}{h_j}$, and $|\xi_i - z_i^{(0)}| = \frac{t_i r_i}{h_i}$, $i \neq j$ This proves (i).

(ii) We have by (i), $R_h\ (r,h,z^{(0)}) \leq 2R_{h_1}1,\ h_2,\dots,\ h_n(r,h,z^{(0)})$

$$\leq \dots \leq 2^{h_1}R_O,\ h_2,\dots,\ h_n(r,h,z^{(0)})$$
$$\leq \dots \leq 2^{\|h\|}\,R_{\bar{0}}(r,h,z^{(0)}).$$

$$\text{But } R_h\,(r,h,z^{(0)}) = \max_{\substack{k \leq N \\ |z_i - z_i^{(0)}| = r_i}} \left[\frac{|f^{(k)}(z)|}{k!}\right]$$

$$\text{and } R_{\bar{0}}\,(r,h,z^{(0)}) = \max_{k \leq N}\left[\frac{|f^{(k)}(z^{(0)})|}{k!}\right]. \text{ Hence the result.}$$

Theorem 5. If an entire function f is of boundd index, then for each $r \gg \bar{0}$ there exist an $N=N(r)\ \varepsilon\ I^n$ and a constant $M=M(r) > 0$ such that for each $z^{(O)}\ \varepsilon\ C^n$ there exists a constant $k=k(z^{(0)})\varepsilon\ I^n$ with $k \leq N$ and

$$\max_{|z_i - z_i^{(0)}| = r_i} |f^{(k)}(z)| \leq M\,|f^{(k)}(z^{(O)})|.$$

Proof. For $r \gg \bar{O}$, $z^{(0)}\varepsilon\ C^n$, and an entire function f, we define $M_p(f,z^{(O)},r) =$

$$\max_{|z_i - z_i^{(0)}| = r_i} |f^{(p)}(z)|, \; p \in I^n.$$

Let f be an entire function of b.i. with an index L. For $r >> \bar{O}$ we choose $h \in I^n_+$ such that $\frac{r_i}{h_i} \le \frac{1}{4(L_i+1)}$. Now we take $N = N(r) = L$ and $M=M(r) = 2^{\|h\|}N!$. For $z^{(0)} \in C^n$ we take $k = k(z^{(0)}) \in I^n$ to be an index of f at $z^{(0)}$. Clearly, $k \le L = N$. By the lemma (ii) we have,

$$\max_{\substack{k \le N \\ |z_i - z_i^{(0)}| = r_i}} \left[\frac{|f^{(t)}(z)|}{t!}\right] \le 2^{\|h\|} \max_{t \le N} \left[\frac{|f^{(t)}(z^{(0)})|}{k!}\right] = 2^{\|h\|} \frac{|f^{(k)}(z^{(0)})|}{k!}$$

Hence, $M_k(f,z^{(0)}, r) = \max\limits_{|z_i - z_i^{(0)}| = r_i} |f^{(k)}(z)| \le$

$$N! \max_{\substack{t \le N \\ |z_i - z_i^{(0)}| = r_i}} \left[\frac{|f^{(t)}(z)|}{t!}\right] \le 2^{\|h\|} N! \left[\frac{|f^{(k)}(z^{(0)})|}{k!}\right]$$

$$\le M |f^{(k)}(z)^{(0)})|.$$

Thus for each $r >> \bar{O}$ there exist an $N=N(r) \in I^n$ and a constant $M = M(r) > O$ such that for each $z^{(O)} \in C^n$ there exists a $k = k(z^{(0)}) \in I^n$ with $k \le N$ and

$$\max_{|z_i - z_i^{(0)}| = r_i} |f^{(k)}(z)| \le M |f^{(k)}(z^{(0)})|.$$

A Problem. We now ask the question: Is the condition stated in theorem 5 also sufficient for an entire function to be of b.i. ? Though the answer is "yes" for functions in one variable [1], we do not know the answer for functions in several complex variables. However, assuming that the condition is also suficient for functions of several variables, we try to obtain a proof for that. The proof is incomplete since we have been able to deal with only one case out of two cases for the proof.

For the sufficient part we take $r = \bar{2}$. Then there exist an $N=N(\bar{2}) \in I^n$ and a constant $M = M(\bar{2}) > O$ such that for each $z \in C^n$ there exists a $k=k(z) \in I^n$ with $k \le N$ and $M_k(f,z,\bar{2}) \le M|f^{(k)}(z)|$. Also we can choose an $m \in I^n_+$ such tht $N!\, M 2^{-\|m\|} \le 1$. We propose to show that f is of b.i.

Case 1) Let $p \in I^n$ such that $p \ge m + N$.

We take any point $z^{(0)} \in C^n$. Then there exists a $t = t(z^{(0)}) \in I^n$, $t \le N$ such that $M_t(f,z^{(0)}, \bar{2}) \le M \| f^{(t)}(z^{(0)})|$.

For any entire function g, we have by Cauchy's inequality,

$$|g^{(q)}(z)| \leq \frac{q!}{R^q} \max_{|\xi_1 - z_1| = R_1} |g(\xi)|, \text{ for } q \varepsilon\ I^n \text{ and any } R >> \overline{O}.$$

If we take $g(z) = f^{(t)}(z)$ and $R = \overline{2}$ we get,

$$\frac{|f^{(t+q)}(z)|}{q!} \leq 2^{-\|q\|} \max_{|\xi_i - z_i| = 2} |f^{(t)}(\xi)| = 2^{-\|q\|} M_t(f, z, \overline{2})$$

$$\text{Hence, } \frac{|f^{(p)}(z^{(0)})|}{p!} \leq \frac{|f^{(t+p-t)}(z^{(0)})|}{(p-t)!} \leq 2^{-\|p-t\|}.M_t(f, z^{(0)}, \overline{2})$$

$$\leq 2^{-(\|p\| - \|t\|)}.\ M|f^{(t)}(z^{(0)})| \leq 2^{\|t\| - \|p\|}.\ M|f^{(t)}(z^{(0)})|$$
$$\leq 2^{\|N\| - \|p\|}.\ M|f^{(t)}(z^{(0)})| \leq 2^{-\|m\|}.\ M|f^{(t)}(z^{(0)})|$$
$$\leq 1/N!\ |f^{(t)}(z^{(0)})|$$

$\leq \frac{1}{t!} |f^{(t)}(z^{(0)})|$ Thus, $\frac{1}{p!} |f^{(p)}(z^{(0)})| \leq \frac{1}{t!} |f^{(t)}(z^{(0)})|$ for a $t \leq N$, when $p \geq m+N$.

Case 2. When p is such that neither $p \leq m+N$ nor $p \geq m+N$ we observe that the values of p for which $\leq m+N$ and $\|p\| \leq \|m+N\|$ are finite in number. Hence we need to consider only those values of p for which neither $p \leq m+N$ nor $p \geq m+N$ and $\|p\| > \|m+N\|$. For such a $p\ \varepsilon\ I^n$, there exist at least one j and one l ($1 \leq j, l \leq n$) such that $p_j > m_j + N_j$ and $p_l < m_l + N_l$. Now the problem is : How to deal with such values of p ?

References

[1] Fricke, G. H. : A characterization of functions of bounded index. Indian Jour. of Math. Vol 14, No.3 (1972). pp 207-212.

[2] Krishna, J.G. and Shah, S. M. : Functions of bounded indices in one and several complex variables. Math. Essays dedicated to A. J. Macintyre. Ohio Univ. Press, Athens, Ohio (1970) pp 223-235.

[3] Lepson, B. : Differential equations of infinite order, Hyperdirichlet series and entire functions of bounded index. Proc. Sympo. Pure Math, Vol 11, Amer, Math. Soc (1968) pp 298-307.

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A TYPE OF QUARTER-SYMMETRIC METRIC CONNECTION ON A $(PRS)_n$

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Abstruct : A Riemanian manifold M^n with Levi-Civita connection ∇ along with a pseudo Ricci-symmetric structure and a particular quarter symmetric metric connection $\overline{D}$ has been studied. The relation between the curvature tensor $\overline{R}$ and R, the Ricci tensors $\overline{S}$ and S, the scalar curvatures $\overline{r}$ and r of $\overline{D}$ and ∇ respectively has been derived. Also necessary conditions for the conformal curvature tensor $\overline{C}$ of $\overline{D}$ to be equal to the cônformal curvature tensor C of ∇ are given. In the last section we discussed the prescription $\overline{R} = \phi R$, where ϕ is real function on M.

Introduction : Chaki[1] consider a pseuo Ricci-symmetric manifold $(PRS)_n$ and proved the existence of such structure on a Riemannian menifold. Also he proved that if the scalar curvature r is constant, then it must be zero and if $r \neq O$, then the associated 1-form π must be closed. The object of the paper is to further stydy $(PRS)_n$ with a particular type of quarter symmetric metric connection.

Consider an n-dimensional Riemannian manifold M^n with a metric tensor g and its L.C. connection ∇. Also consider the Pseudo Ricci symmetric structure [1]

1.1) $(\nabla_X S)(Y,Z) = 2\pi(X)S(Y,Z) + \pi(Y)S(X,Z) + \pi(Z)S(X,Y)$

which has the properties

1.2) $\pi(LX) = 0$

1.3) $S(X,\rho) = 0$

1.4) $d\pi(X,Y) = 0$ if $r \neq 0$

where π is a 1-form, and ρ is a particular vectorfield such that $\pi(X) = g(X,\rho)$, and L is the symmetric endomorphisms of the tangent space at each point of (M^n,g) which is such that $S(X,Y) = g(LX,Y)$

Further consider the quarter symmetric-metric connection $\overline{D}$ [2] suh that its torsion tensor $\overline{T}$ is given by

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1.5) $\overline{T}(X,Y) = \overline{D}_X Y - \overline{D}_Y X - [X,Y] = \pi(Y)LX - \pi(X)LY$

For simplicity we shall write $(PRSQ)_n$ for a Riemannian manifold endowed with a $(PRS)_n$ structure and the above quarter symmetric metric connection $\overline{D}$.

2. Preliminaries :

Let us define $\overline{D}$ as follows

2.1) $\overline{D}_X Y = \nabla_X Y + H(X,Y)$

It is known that

2.2) $(\overline{D}_X g)(Y,Z) = Xg(Y,Z) - g(\overline{D}_X Y,Z) - g(Y,\overline{D}_X Z) = 0$

By virture of (2.1) we have

2.3) $g(H(X,Y),Z) + g(H(X,Z),Y) = 0$

From (2.1) and (1.5) we can get

2.4) $H(X,Y) - H(Y,X) = \pi(Y)LX - \pi(X)LY$

Next making cyclic peramutation of X,Y,Z in (2.3) we have

2.5) $g(H(Y,Z,),X) + g(H(Y,X),Z) = 0$

2.6) $g(H(Z,X),Y) + g(H(Z,Y),X) = 0$

Substracting (2.5) and (2.6) from (2.3) we get by virtue of (2.4)

$g(H(Y,Z),X) = 2\pi(Z)S(X,Y) - 2\pi(X)S(Y,Z)$

Therefore we can have by virtue of (2.1)

2.7) $\overline{D}_x Y = \nabla_X Y + \pi(Y)LX - S(X,Y)\rho$

Let $\overline{R}(X,Y,Z) = \overline{D}_X \overline{D}_Y Z - \overline{D}_Y \overline{D}_X Z - \overline{D}_{[X,Y]} Z$ denotes the curvature tensor of the connection $\overline{D}$ then from (2.7) we can have

2.8) $$\begin{aligned}\overline{R}(X,Y,Z) = \overline{\ R}(X,Y,Z) &+ (\nabla_X\pi)(Z)LY - (\nabla_Y\pi)(Z)LX + \pi(Z)[\nabla_X L)(Y) - \\ &(\nabla_Y L)(X)] - [(\overline{\nabla}_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - S(Y,Z) \\ &[\nabla_X\rho + \pi(\rho)LX] + S(X,Z)\ [\nabla_y\rho + \pi(\rho)LY]\end{aligned}$$

But from (1.1) we have

2.9) $(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \pi(X)S(Y,Z) - \pi(Y)S(X,Z)$

Hence (2.8) can be written in the form

2.10) $\overline{R}(X,Y,Z) = R(X,Y,Z) + [(\nabla_X\pi)(Z) + \pi(X)\pi(Z) + 1/2\pi(\rho)S(X,Z)]LY - [(\nabla_Y\pi)(Z) + \pi(Y)\pi(Z) + 1/2\pi(\rho)S(Y,Z)]LX + S(X,Z)[\nabla_Y\rho + \pi(Y)\rho + 1/2\pi(\rho)LY] - S(Y,Z)[\nabla_X\rho + \pi(X)\rho + 1/2\pi(\rho)LX]$

Let us put

2.11) $\lambda(X,Y) = (\nabla_X\pi)(Y) + \pi(X)\pi(Y) + 1/2\pi(\rho)S(X,Y) = g(QX,Y)$

It is clear that λ is symmetric if and ony if π is closed i.e. $r \neq 0$.

Now from (2.10) and (2.11) we have

2.12) $\overline{R}(X,Y,Z) = R(X,Y,Z) + \lambda(X,Z)LY - \lambda(Y,Z)LX + S(X,Z)QY - S(Y,Z)QX$

Contracting (2.12) with respect to X yields

2.13)a) $\overline{S}(Y,Z) = S(Y,Z) + \lambda(LY,Z) - r\lambda(Y,Z) + \lambda(Y,LZ) - aS(Y,Z)$

where $a = \text{trace } Q = \text{div}\pi + ((r+2)/2)\ \pi(\rho)$

If λ is symmetric then it is easy to see that $\lambda(X,LY) = \lambda(LX,Y)$. Therefore (2.13)a can be reduced to the form

2.13)b) $\overline{S}(Y,Z) = S(Y,Z) + 2\lambda(LY,Z) - r\lambda(Y,Z) - aS(Y,Z)$

From which we can have

2.14) $\overline{L}X = LX + 2Q(LX) - r(QX) - aLX$

Contracting (2.13)b we get

2.15)a $\overline{r} = r - 2ar + 2(\nabla_X\pi)(LX) + \pi(\rho)S(X,LX)$

Now using (1.1), (1.2) and (1.3) on $(\nabla_X\pi)(LY)$ we can have

2.16) $(\nabla_x\pi)\ (LY) = \pi(\rho)\ S(X,Y)$

Hance (2.15)a reduces to the form

2.15)a $\overline{r} = r[1 - 2a - 2\pi(\rho)] + \pi(\rho)S(X,LX)$

Now we can state

Lemma 2.1) On a $(PRSQ)_n$ of nonzero scalar curvature, the curvature tensor $\overline{R}$, Ricci tensor $\overline{S}$ and the scalar curvature $\overline{r}$ of the connection $\overline{D}$ are given by (2.12), (2.13)b (2.15)b respectively.

From (2.11) and (2.13) we can see that $\overline{S}$ is symmetric iff λ is symmetric i.e. iff π is closed.

Thus we can state

Theorem 2.1) On a $(PRSQ)_n$ the Ricci tensor of $\overline{D}$ is symmetric if and only if π is colsed.

From 2.11) also we can state

Theorem 2.2) On a $(PRSQ)_n$ the necessary and sufficient condition for $\overline{R} = R$ is

2.17) $\lambda(X,Z)LY - \lambda(Y,Z)LX + S(X,Z)QY - S(Y,Z)QX = 0$.

From now on we will consider the case $r \neq O$, i.e. π is closed.

Using (2.11) and (2.16) we can get by virtue of (1.3)

2.18) $\lambda(X,L\rho) = \lambda(LX,\rho) = O$

Now contracting (2.17) with respect to X we can have

2.19) $2\lambda(Y,LZ) - r\lambda(y,LZ) - aS(Y,Z) = O$

Let $Z = \rho$, then we have using (1.3) and (2.18)

$r\lambda(Y,\rho) = O$

Since $r \neq O$, we have $\lambda(X,\rho) = O$.

Thus we can state

Corollary 2.1) On a $(PRSQ)_n$ the necessary condition for $\overline{R} = R$ is $\lambda(X,\rho) = O$.

Note that $\lambda(X,\rho) = O$ is sufficient condition for $\overline{R}(\rho,X,Y) = R(\rho,X,Y)$.

Hence we can state

Theorem 2.3) On a $(PRSQ)_n$ the necessary and sufficient condition for $\overline{R}(\rho,X,Y) = R(\rho,X,Y)$ is $\lambda(X,\rho) = O$

3. CONFORMAL CURVATURE TENSOR.

The conformal curvature tensor $\overline{C}$ of $\overline{D}$ is given by

$$3.1)\ \overline{C}(X,Y,Z) = \overline{R}(X,Y,Z) - \frac{1}{n-2}[\overline{S}(Y,Z)X - \overline{S}(X,Z)Y + g(Y,Z)\overline{L}X - g(X,Z)\overline{L}Y]$$

$$+ \frac{\overline{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]$$

By virtue of (2.12), (2.13)b and 2.15)b we can have

$$3.2)\ \overline{C}(X,Y,Z) = C(X,Y,Z) + \lambda(X,Z)LY - \lambda(Y,Z)LX + S(X,Z)QY - S(Y,Z)QX$$

$$-\frac{1}{n-1}[2\lambda(LY,Z)X - r\lambda(Y,Z)X - aS(Y,Z)X - 2\lambda(LX,Z)Y + r\lambda(X,Z)Y$$

$$+ aS(X,Z)Y - g(X,Z)Q(LY) + rg(X,Z)L(QY) - g(x,z)L(QY) + ag(X,Z)LY$$

$$+ g(Y,Z)Q(LX) - rg(Y,Z)QX + g(Y,Z)L(QX) - ag(Y,Z)LX]$$

$$+ \frac{1}{(n-1)(n-2)}[-2ra - 2r\pi(\rho) + \pi(\rho)S(X,LX)][g(Y,Z)X - g(X,Z)Y]$$

Now let $\overline{C} = C$ in (3.2) and contracting with respect to X we can have

3.3) $\lambda(LY,Z) - S(QY,Z) = [-r\pi(\rho) + \frac{1}{2}\pi(\rho)S(X,LX) - S(QX,X)]\ g(Y,Z)$

Contracting again (3.3) we can get

3.4) $(n-1)\ S(QX,X) = O$

Thus we can state

Theorem 3.1) on a $(PRSQ)_n$, $n > 1$, the necessary condition for $\overline{C} = C$ is LX orthogonal to QX.

Using (2.11) on (3.4) we can have

3.5) $S(X,\nabla_X\rho) + \frac{1}{2}\pi(\rho)S(X,LX) = O$

By virtue of (1.1) and (2.16) we get

$\pi(\rho)\ [-r + \frac{1}{2}S(X,LX)] = O$

Since $\pi(\rho) \neq O$, we have

3.6) $S(X,LX) = 2r$

Thus we can state

Corollary 3.1) : On a $(PRSQ)_n$, $n > 1$, the necessary condition for $\overline{C} = C$ is $S(X,LX) = 2r$.

Now let $\overline{R} = O$, then from (2.12) we have

3.7) $R(X,Y,Z) = \lambda(Y,Z)LX - \lambda(X,Z)LY - S(X,Z)QY + S(Y,Z)QX$

Contracting with respect to X we get

3.8) $S(Y,Z) = r\lambda(Y,Z) - 2\lambda(LY,Z) + aS(Y,Z)$

Containing again (3.8) we get

3.9)a) $r = 2ra - 2\lambda(LX,X)$

Using (2.16) and the definition of $\lambda(LX,Y)$ on (3.9)a we get

3.9) b) $r = 2ra + r\pi(\rho) - \frac{1}{2}(\rho)S(X,LX)$

or

3.10) $a = \frac{1}{2} - p(r) + \frac{1}{4r}p(r)S(X,LX)$

Let C = O then we have

3.11) $R(X,Y,Z) - \frac{1}{n-2}[S(Y,Z)Y - S(x,z)y + g(Y,Z)LX - g(X,Z)LY]$

$$+\frac{r}{(n-1)\,(n-2)}[g(Y,Z)X-g(X,Y)Z]=0$$

Substituting from (3.7), (3.8) and (3.10) in (3.11) and contracting with respect to X we get

3.12) $r\lambda(Y,Z)-2(n-2)\,\lambda(LY,Z)-(n-2)\pi(\rho)S(Y,Z)+\frac{(n-2)}{2}\pi(\rho)S(LY,Z)=0$

Let $Z=\rho$ in (3.12) then by virtue of (1.3) and (2.18) we get

$r\lambda(Y,\rho)=O.$

Since $r\neq O$, we have $\lambda(y,\rho)=O.$

Thus we can state

Theorem 3.2) On a $(PRSQ)_n$ if $\overline{R}=O$, then the necessary condition for the conformal curvature tensor C to be flat is that QX is orthogonal to ρ'.

4. Prescribed Curvature Tensor of $\overline{D}$.

Let us consider the prescription of the curvature tensor $\overline{R}$ in which $\overline{R}=\phi R$, where ϕ is a real function on M. Then from (2.12) and by a proper contraction we can prove the following lemma.

Lemma 4.1) On a $(PRSQ)_n$ if $\overline{R}=\phi R$, ϕ is a real function on M, then we have

4.1) $(\phi-1)R(X,Y,Z)=\lambda(X,Z)LY-\lambda(Y,Z)LX+S(X,Z)QY-S(Y,Z)QX$

4.2) $(\phi-1)S(Y,Z)=2\lambda(Y,LZ)-r\lambda(Y,Z)-aS(y,z)$

4.3) $(\phi-1)r=-2r\pi(\rho)+\pi(\rho)S(X,LX)-2ra.$

Now let $Z=\rho$ in (4.2) we can have by using (1.3) and (2.18)

$r\lambda(Y,\rho)=O$

There since $r\neq O$ by assumption we have

4.4) $\lambda(X,\rho)=O$

Let $Y=\rho$ in (2.11) then we have

4.5) $(\nabla_X\pi)(\rho)=-\pi(X)\pi(\rho)$

Now let the projective curvature tensor $\overline{W}$ of $\overline{D}$ given by

4.6) $\overline{W}(X,Y,Z)=\overline{R}(X,Y,Z)-\frac{1}{n-1}[\overline{S}(Y,Z)X-\overline{S}(X,Z)Y]$

Using the prescription $\overline{R}=\phi R$ on (4.6) and adding and substracting each of R(X,Y,Z),

$\frac{1}{n-1}$ S(Y,Z)X and $\frac{1}{n-1}$ S(X,Z)Y we get

$$\overline{W}(X,Y,Z) = W(X,Y,Z) + (\phi-1)R(X,Y,Z) - \frac{1}{n-1}[(\phi-1)S(Y,Z)X - (\phi-1)S(X,Z)Y]$$

Using (4.1) and (4.2) on the above equation, then let $Z = \rho$ we can have by virtue of (4.4) and (2.18) that

4.7) $\overline{W}(X,Y,\rho) = W(X,Y,\rho)$

Also using (4.1) and (4.2) on (4.6) and let $Z = \rho$, then by virtue of (4.4) and (2.18) we can have

4.8) $\overline{W}(X,Y,\rho) = O$.

Thus we can state

Theorem 4.1) On a $(PRSQ)_n$ if $\overline{R} = \phi R$, ϕ is a real function on M. then we have

$\overline{W}(X,Y,\rho) = W(X,Y,\rho) = O$.

Now let $Z = \rho$ in (4.1), then by virtue of (4.4) and (2.18) we have

$(\phi-1)\, R(X,Y,\rho) = O$.

Thus we can state

Theorem 4.2) On a $(PRSQ)_n$ if $\overline{R} = \phi R$, ϕ is a real function on M, and $\phi > 1$, then $R(X,Y,\rho) = O$.

References

1. Chaki, M. C. : On Pseudo Ricci-symmetric manifolds, Bulg. J. Phys. 15(1988). 6, 526 - 531.

2. Mishra, R. S. : Structure on a differentiable manifold and their applications. Chandrama Prakashan, Allahabad, 1984.

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IDEALS OF A G-REGULAR RING

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1. A regular ring R is G-regular iff the annihilator of its centre is the zero ideal i.e. R is G-regular iff Ann(C(R)) in R = (O). This paper is devoted to the study of the ideals of G-regular rings.

2. We will prove a theorem which will show that the G-regularity is a hereditary property for ideals of a ring. But before we enter into the proof of this theorem we will prove some lemmas, which will be useful to us to prove the theorem.

Lemma 2.1. The ideals of a regular ring R are regular.

Proof : Let I be any both sided ideal of R and let a ε I. As R is regular, there exists a′ ε R such that aa′a = a. Put b = àeaàeH. Then b ε I. Also aba = aa′aa′a = aa′a = a. Thus I is a regular ring.

Lemma 2.2 Let I be any ideal of the regular ring R. Then any ideal J of I is also an ideal of R.

Proof : Let a ε J and r ε R, then ar,ra ε I. By lemma 2.1 I is a regular ring. Then ∃ X ε I such that arxar = ar we get ar = a (rxar) ε J, as rxar ε I and J is an ideal of I, a(rxar) ε J. Similarly, ra ε J.

Lemma 2.3 If I is an ideal of a ring or R, then I ∩ C(R) ⊆ C(I), where C(R) and C(I) are the centres of R and I respectively.

Proof : Let p ε I ∩ C(R) and q be any element of I, then q ε I ⊆ R = pq = qp as p ε C(R). Then p ε I and commutes with every element of I. Thus I ∩ C(R) ⊆ C(I).

Lemma 2.4. Let I be an ideal of a ring R and e ε C(I), where e^2 = e, then e ε I ∩ C(R).

Proof : Let e^2 = e ε C(I) ⊆ I. Let p ε R, then ep,pe ε I, which shows ep+pe ε I. But e ε C(I). Thus we have

e(ep+pe) = (ep+pe)e

=> ep+epe = epe + pe => ep = pe => e ε C(R).

From Lemma 2.3 and 2.4 naturally the question arises wheather the quality C(I) = I

$\cap$ C(R) hold for an arbitrary ring R, where I is a both sided ideal of R, and C(I) and C(R) denote the centres of I and R respectively.

Now we give the following example to show that the equality mentioned above does not hold in general.

Let $R = \left\{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \middle/ a, b \in J\right\}$

Then R is a ring.

Let $I = \left\{\begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix} \middle/ a \in J\right\}$

Then I is an ideal of R.

I is a commutative subring of R and therefore C(I) = I.

Now $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in I = c(I)$.

If possible, let $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in I = c(R)$.

Now $\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \in R$.

Hence $\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$

That is $\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}$ contradiction as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in c(R)$.

Thus we have C(I) $\neq$ I $\cap$ C(R).

Lemma 2.5.

If M is a subring of a ring R and R is embedded in a ring R′, then M is subring of R′.

The proof is obvious.

Lemma 2.6.

Any minimal ideal (when exists) of a G-regular ring is G-regular.

Proof : Let R be a G-regular ring andM be a minimal ideal of R.

As R is G-regular, R can be embedded in a regular ring R′ with 1 [3].

Hence M is a subring of R′ by lemma 2.5. Again a regular ring D with 1 can be constructed having the properties

i) $C(M)$ can be embedded in D

ii) $\mathrm{Ann}(C(M)) = (O)$ in D. [3].

We observe that if M is a minimal ideal, it is a simple ring. Also by lemma 2.1. it is regular. So we have M has atmost O and 1 as its central idempotents [3].

But $C(M) \neq (O)$. Indeed $C(M) = (O) \Rightarrow \mathrm{Ann}\ (C(M))$ in $D \neq D \neq (O)$.

Contradiction. Hence M has an identity. Therefore $\mathrm{Ann}(C(M))$ in $M = (O)$. Hence M is G-regular.

We now prove the following theorem which appears interesting to us.

Theorem 2.7.

Every ideal of a G-regular ring is G-regular.

Proof : We recall that the set of all non-null ideals of a ring form a poset with respect to usual set inclusion. Denote it by $P_o(I)$. Evidently the minimal ideals of R (if exists) are the minimal elements of $P_o(I)$. We have proved that they are G-regular. Suppose G-regularity holds for all $I \underset{\neq}{<} J$. We now want to show J is G-regular. By 2.1., J is regular.

First we will show that J contains at least two central indepotents.

Indeed, any $I \underset{\neq}{<} J$ always contains at least one non-zero central idempotent say e, otherwise $C(I) = (O) \Rightarrow \mathrm{Ann}\ (C(I)$ in $I = I$. As I is G-regular, we arrive at a contradiction.

I is also an ideal of J and by 2.4., this idempotent e is also in $C(J)$. Hence J contains at least one non-null central idempotent. Put $T = \mathrm{Ann}\ (C(J))$ in J. Then T is a both sided ideal in J. But $T \neq J$, otherwise $T = J$ shows $C(J) = (O)$.

But we have just now proved J contains at least one non-zero central imdempotent.

Thus T is a proper both sided ideal in J. We have to show $T = (O)$. By induction hypothesis T is G-regular.

Let $f \in T$, that is, $f \in \mathrm{Ann}(C(J))$ in $J \Rightarrow f \in J$ and $f(C(J)) = (O)$ — (1).

Let h be any idempotent in $C(T)$. Then by lemma 2.4, $h \in C(J)$. Hence by (1) $fh = o$ — (2).

Now let x be any element of $C(T)$. As $C(T)$ is regular, $\exists\ g \in C(T)$ where $g^2 = g$ and $gx = x$. Hence. $fx = fgx = O$, by (2), as $g \in C(J)$, since T is an ideal of J.

Hence $f(C(T)) = (O)$.

But by induction hypothesis, $\mathrm{Ann}(C(T)) = (O)$ in T. Hence $f = O$. Thus T contains only one idempotent, viz., O that is $\mathrm{Ann}(C(J)) = (O)$ in J. Hence, by induction, every both sided ideal of R is G-regular.

On the other hand we have,

Theorem 2.8

Let R be a regular ring having at least one non-zero proper both sided ideal. If every proper both sided ideal of R is G-regular, then R is G-regular.

Proof : Let I = Ann(C(R)) in R, our aim is to show I = (O).

Let J be any proper non-null both sided ideal in R (the existence of such an ideal is guaranted by hypothesis).

By hypothesis, J is G-regular.

Now J being G-regular, C(J) $\neq$ (O) otherwise C(J) = (O) =>

Ann(C(J)) in J = J $\neq$ (O).

Thus $\exists$ e ε C(J) where $e^2 = e \neq O$

which show e = e.e ε JC(J).

Now J$\subseteq$R, and C(J)$\subseteq$ C(R), [by lemma 2.4] which shows e ε RC(R). i.e. RC(R) $\neq$ (O) — (1)

Thus we concluded that I $\neq$ R - (2)

Otherwise I $\neq$ R => Ann(C(R)) in R = R.

i.e. RC(R) = O contradiction (1).

Thus I is a proper ideal of R. By hypothesis I is G-regular. Let $e^2 = e$ ε I. => e(C(R)) = (O) -- (3), as I = Ann(C(R)) in R. Let $g^2 = g$ ε C(I). By lemma 2.4, g ε C(R). Hence eg = O by (3).

Thus for every idempotent g in C(I), eg = O - (4). But I being an ideal of R, C(I) is regular. Let a e C(I), then $\exists$ $p^2 = p$ ε C(I) such that pa = a. Now ea = epa = O by (4).

Hence e ε (C(I) = (O) => e ε Ann(C(I) in I. Thus G-regularity of I shows e = O. Hence I contains only one idempotent viz. O. Let a ε I, but I being regular $\exists$ X ε I such that axa = a => ax is an idempotent. Hence ax = O => a = O.

Therefore I = (O) and R is G-regular.

It should be noted that the above theorem is not true if the regular ring R has only one proper ideal viz. (O) which is obviously G-regular. Indeed, in this case the original ring turns to be a simple regular ring and it is proved that such a ring may or may not be G-regular [3]. Also, there are non-regular rings whose peoper ideal is G-regular.

Now naturally the following question arises : If every proper ideal of a ring R is regular, will the ring R be regular? The result is not yet completely solved. However, we have solved this problem for some particular cases.

Proposition 2.9

If R be a ring, in which every proper ideal is regular and has a non-zero central idempotent e where $Re \neq R$, then R is regular.

Proof : As e is central, Re is a both sided ideal in R. Put $I = Re$. Let $J = \{r - re \,/\, r \,\varepsilon\, R\}$ Then J is a both sided ideal in R.

Also $p \,\varepsilon\, I \cap J \Rightarrow p = ae$ for some $a \,\varepsilon\, R$ and $p = s - se$, for some $s \,\varepsilon\, R$.

Now $pe = ae^2 = ae = p$

and $p = pe = se - se^2 = se - se = o$.

Therefore $I \cap J = (O)$.

Also $r \,\varepsilon\, R = r = re + (r-re)$

$\Rightarrow R = 1 + J$.

Let a R, then $a = a_1 + a_2$, where $a_1 \,\varepsilon\, I$, $a_2 \,\varepsilon\, J$. But by hypothesis $I \neq R$. Also $e \neq O$ shows $J \neq R$. Hence I, J are regular rings, by hypothesis.

Suppose $a_1x_1a_1 = a_1$ where $x_1 \,\varepsilon\, I$

and $a_2x_2a_2 = a_2$ where $x_2 \,\varepsilon\, J$.

Put $x = x_1 + x_2$.

Then we have

$axa = (a_1+a_2)(x_1+x_2)(a_1+a_2)$

$= (a_1x_1+a_2x_1+a_1x_2+a_2x_2)(a_1+a_2)$

As $a_1 \,\varepsilon\, I$, $x_2 \,\varepsilon\, J$, we have $a_1x_2 \,\varepsilon\, I,J$.

But as $I \cap J = O$, we have $a_1x_2 = O$.

Similarly $a_2x_1 = O$.

Thus we are left with $axa = (a_1x_1+a_2x_2)\ (a_1+a_2)$

$= a_1x_1a_1+a_2x_2a_1+a_1x_1a_2+a_2x_2a_2$

$= a_1x_1a_1+a_2x_2a_2$

$= a_1+a_2 = a$.

Proposition 2.10

If R is ring, which is not finitely generated and every proper both sided ideal of R is regular, then R is also regular.

Proof : If possible let R is not regular. Then $\exists$ an element $a \,\varepsilon\, R$ for which we can not find any x in R satisfying $axa = a$. - (1).

Let us consider the principal both sided ideal (a). Evidently $a \,\varepsilon\, (a)$. Then $(a) = R$.

For otherwise $(a) \neq R \Rightarrow (1)$ is satisfied. Contradiction as R is not finitely generated. Hence R is regular.

Rererences

1. Birkhoff, G. : Lattice theory, Am. Math, Soc. Pub XXV, 1968.
2. Burton D. M. : A first course in Rings and Ideals. Additon-Wesley publishing Comp.
3. Ghosh, J. : Journal of Pure Maths. C.U. Vol.-5,1985-86.
4. Herstein, I. N. : Non Commutative rings, The Carus Mathematical Monographs, Number Fifteen.
5. Jacobson, N. : The Theory of rings, Amer, Math. Soc. Surveys No. 2, New York 1943.
6. Kaplansky, I. : Canad. Math. Bull. Vol.14, No. 4, 1971.
7. Kaplansky, I. : Any orthocomplemented modular latice is a continuous geometry, Ann Math.61(1955), 524-541.
8. Koh, K. : Canad. Math. Bull Vol.17, No.2, 1974.
9. Kurosh, A. G. : Lectures in General Alegebra, London Permagon Press 1965.
10. Mcdoy, N. H. : Theory of rings, The Macmillan Company, New York, 1964.
11. Neumann, J. V. : Continuous Geometry, Princeton University Press 1960.
12. Skornjakov, L. A. ; Dedekindovi Strukturi s dopolneniamu e regularnie koltsa. Fuzuko Mathematicheskoe Luteraturi Moskba 1961 (In Russian).
13. Waddel, M. C. : Properties of regular ring, Duke Math. J. J 19(1952), 623-627.

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FIXED POINT THEOREMS IN SOME TOPOLOGICAL SPACES

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Abstract. In this paper an attempt has been made to find out some sufficient conditions for self mapping in certain topological spaces which ensure the existence of fixed point of the mapping considered.

Introduction. Just as the classical contraction mapping theorem of S.Banach is the basis to ensure the existence of fixed point for any self mapping in metric space, similarly the theorems due to Schauder [9] and Tychonoff [11] are backbones to prove fixed point theorem for non-expansive mapping in locally convex linear topoligcal space.

In the first section, we have considered the family $\mathcal{H}$ given by

$\mathcal{H}$ = {h / h : $(\mathbb{R}^+)^3 \longrightarrow \mathbb{R}^+$ is upper semicontiuous and non-decreasing in each coordinate variable}

where $\mathbb{R}^+$ stands for the set of all non-negative real numbers and have considered Hausdorff topological spaces to study two theorems on fixed point. In the second section some results on fixed point in pseudocompact Tychonoff space using rational expression have been studied. The final section contains two fixed point theorems respectively for non-expansive and asymptotically non-expansive mapping of a weakly compact convex subset of a locally convex Hausdorff linear topological space.

1. Fixed Point Theorems in Hausdorff Space

Theorem-1.1 : Let X be a Hausdorff space and $T:X \longrightarrow X$.Let $f:X \times X \longrightarrow \mathbb{R}^+$ be a continuous mapping satisfying conditions

1] $f(x,y) = 0$ if and only if $x = y$ for every $x,y \in X$,

ii] there exists an $h \in \mathcal{H}$ such that for every $x,y \in X$, $x \neq y$

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$$f(Tx,Ty) < h[f(x,y),\ f(x,Tx),\ f(y,Ty),$$

where h satisfies $g(t) = h(t,t,t) < t$ for all $t \in \mathbb{R}^+ - \{0\}$; and if for some $x_0 \in X$ the sequence of iterates $\{T^n x_0\}$ has a subsequence converging to some $u \in X$ and T is continuous at u, then u is the unique fixed point of T. Moreover, the sequence of iterates $\{T^n x_0\}$ itself converges to u.

Proof : In order to prove the above theorem we need the following

Lemma :- [[8], [10]] :- Suppose that $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a non-decreasing upper semi-continuous mapping. Then for every $t > 0$, $\phi(t) < t$ if and only if $\lim_{n \to \infty} \phi^n(t) = 0$, where ϕ^n denotes the composition of ϕ with itself n times.

Let $x_o \in X$ be any point. We now construct a sequence $\{x_n\}$ such that $x_n = T^n x_o$.

Let us write $d_n = f(Tx_n, Tx_{n+1})$, $\forall\ n \in \mathbb{N} \cup \{0\}$, where IN stands for the set of all natural numbers.

We shall first show that $d_{n+1} < d_n$, $\forall\ n \in \mathbb{N} \cup \{0\}$.

Let us assume that $t = d_{n+1} \geq d_n$ and $t > 0$. Now

$$t = f(Tx_{n+1}, Tx_{n+2}) < h[f(x_{n+1}, x_{n+2}),\ f(x_{n+1}, Tx_{n+1}),\ f(x_{n+2}, Tx_{n+2})]$$
$$\leq h[t,t,t] < t \text{ —— which leads to a contradiction.}$$

Hence the assertion is true. Thus $\{d_n\}$ is a strictly decreasing sequence of non-negative real numbers, Now,

$$d_1 = f(Tx_1,Tx_2) < h[f(x_1,x_2),\ f(x_1,Tx_1),\ f(x_2,Tx_2)] \leq h[d_o,d_o,d_o] = g(d_o).$$

So, in general, we have $d_n < g^n(d_o)$, $\forall\ n \in \mathbb{N} \cup \{0\}$

If $d_o > 0$, then by the lemma above, we have

$\lim_{n \to \infty} g^n(d_o) = 0$ as $g(d_o) < d_o$ i.e. $\lim_{n \to \infty} d_n = 0$.

If $d_o = 0$, then $d_n = 0$ for each n.

Hence in any case, $\lim_{n \to \infty} d_n = 0$ i.e. $\lim_{n \to \infty} f(x_{n+1}, x_{n+2}) = 0$.

Again, the sequence $\{x_n\}$ has a subsequence, say $\left\{x_{n_k}\right\}$ which converges to some u in X. So we have

$$f(u,Tu) = f(\lim_{k \to \infty} x_{n_k}\ T(\lim_{k \to \infty} x_{n_k})) = \lim_{k \to \infty} f\left(x_{n_k}, Tx_{n_k}\right) = 0.$$

Therefore, $u = Tu$.

For uniqueness, let $v(\neq u)\in X$ be such that $Tv = v$. Then we have $f(u,v) = f(Tu,Tv) < h[f(u,v), f(u,Tu), f(v,Tv)] < f(u,v)$ — which is a contradiction.

Hence u is the unique fixed point of T in X.

To complete the proof it remains to show that $x_n \longrightarrow u$ as $n \longrightarrow \infty$, Since both $f\left(u, x_{n_k}\right)$ and $f\left(x_{n_k}, x_{n_{k+1}}\right)$ tend to zero as k tends to ∞, therefore, for a given $\partial > 0$, we can choose $n_k \in \mathbb{N}$ such that $f\left(u, x_{n_k}\right) < \partial$ and $f\left(x_{n_k}, x_{n_{k+1}}\right) < \partial$. Then by condition (ii) we get

$$f(u,x_n) = f(Tu,Tx_{n-1}) < h[f(u,x_{n-1}), f(u,Tu), f(x_{n-1},Tx_{n-1})] \leq h[\alpha,\alpha,\alpha] < \alpha$$

where $\alpha = \max\ [f(u,x_{n-1}), f(x_{n-1}, Tx_{n-1})]$,

i.e. $f(u,x_n)<\max[f(u,x_{n-1}), f(x_{n-1},x_n)]$ (1)

Similarly, we can obtain $f(u,x_{n-1}) < \max[f(u,x_{n-2}), f(x_{n-2},x_{n-1})]$ (2)

Now we choose $n > n_k$ and by the repeated application of (1) and (2) we have $f(u,x_n) < \max[f(u,x_{n-3}), f(x_{n-3}, x_{n-2})] < \ldots..$

$$< \max\left[f\left(u,x_{n_k}\right), f\left(x_{n_k}, x_{n_k+1}\right)\right] < \partial,$$

$$\Rightarrow \lim_{n\to\infty} f(u,x_n) = 0 \Rightarrow f(u,\lim_{n\to\infty} x_n) = 0 \Rightarrow \lim_{n\to\infty} x_n = u,$$

which is as desired. This completes the proof of the theorem.

Theorem-1.2 :- Let X be a first countable Hausdorff space and $T_1,T_2:X\to X$. Let $f:X \times X \to R^+$ be a continuous mapping such that

i] $f(x,y) = 0$ if and only if $x = y$ for every $x,y \in X$;

ii] there exists an $h \in \mathcal{H}$ such that for every $x,y \in X$, $x \neq y$

$$f(T_1x,T_2y)<h[f(x,y), f(x,T_1x), f(y,T_2y)]$$

where h is as in theroem-1.1, and if for some $x_0 \in X$, any one of the sequences of iterates $\{(T_2T_1)^n x_0\}$ and $\{T_1(T_2T_1)^n x_0\}$ has a cluster point $u \in X$ where $x_{2n+1} = T_1\ x_{2n}$ and $x_{2n+2} = T_2x_{2n+1}$ for every $n > 0$ and each of T_1,T_2,T_1T_2 and T_2T_1 is continuous at u then T_1 and T_2 have a unique common fixed point u. Further, the sequence of iterates which has a cluster point u, will itself converge to u.

Proof : We have that

$$f(x_1,x_2) = f(T_1x_0, T_2x_1) < h[f(x_0,x_1),f(x_0,Tx_0),f(x_1,Tx_1)]$$

$$< h[f(x_0,x_1),f(x_0,x_1),f(x_1,x_2)].$$

If $f(x_0,x_1) \leq f(x_1,x_2)$, then $f(x_1,x_2) < h[p,p,p] < p = f(x_1,x_2)$ - which is a contradiction, hence $f(x_0,x_1) > f(x_1,x_2)$ and the repeated arguement shows that $\{f(x_n,x_{n+1})\}$ is a stirctly decreasing sequence of non-negative real numbers which is bounded below, therefore it must converge to some limit. So let $\sigma = \lim_{n\to\infty} f(x_n,x_{n+1})$.

If the sequence of iterates $\{(T_2T_1)^n x_0\}$ has a cluster point $u \in X$, it will yield a convergent subsequence $\left\{x_{n_k}\right\}$ (say), with limit u where n_k 's are even. Therefore, we have

$$\sigma = \lim_{k\to\infty} f(x_{n_k}, T_1 x_{n_k}) = f(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} T_1 x_{n_k}) = f(u, T_1 u) \text{ and}$$

$$\sigma = \lim_{k\to\infty} f(x_{k_k+1}, x_{n_k+2}) = f(\lim_{n\to\infty} T_1 x_{n_k}, \lim_{k\to\infty} T_2 T_1 x_{n_k}) = f(T_1 u, T_2 T_1 u), \text{ since}$$

T_1 and T_2T_1 are continuous at u.

We shall now show that $\sigma = 0$. Suppose that $\sigma > 0$. Then we have $\sigma = f(T_1u,T_2T_1u) < h[f(u,T_1u), f(u,T_1u), f(T_1u,T_2T_1u)] < f(u,T_1u) = \sigma$ which gives rise to a contradiction. Hence $\sigma = 0$. and therefore we have $u = T_1u$ and also

$T_1u = T_2T_1u$. Now $T_2T_1u = T_1u \Rightarrow T_2u = u$.

thus u is a common fixed point of T_1 and T_2.

A similar conclusion can be had for the case of $\{T_1(T_2T_1)^n x_0\}$ using continuity of T_2 and T_1T_2 at u.

Now for uniqueness of u, let $v(\neq u) \in X$ be such that $T_1v = v = T_2v$. Then by condition (ii) we have

$f(u,v) = f(T_1u,T_2v) < h[f(u,v),f(u,T_1u),f(v,T_2v)] < f(u,v)$ — which is absurd. Hence u is the common fixed point of T_1 and T_2.

It now remains to show that any one of the sequences of iterates$\{(T_2T_1)^n x_0\}$ and $\{T_1(T_2T_1)^n x_0\}$ which has cluster point u, will itself converge to u.

Here we observe that $(T_2T_1)^n x_0 = x_{2n}$ and $T_1(T_2T_1)^n x_0 = x_{2n+1}$.

Suppose first that u is a cluster point of the sequence $\{x_{2n}\}$. As X is first countable, there is a subsequence $\left\{x_{2n_k}\right\}$ of the sequence $\{x_{2n}\}$ which converges to u. Since $f(u,x_{2n_k}) \to 0$ and $f(x_{2n_k}, x_{2n_k+1}) \to 0$ as $k\to\infty$, therefore, given $\partial > 0$ we can choose $n_k \in \mathbb{N}$ such that $f(u,x_{2n_k}) < \partial$ and $f(x_{2n_k}, x_{2n_k+1}) < \partial$. Now by condition-(ii), we have

$$f(u,x_{2n}) = f(T_1\, u,\ T_2x_{2n-1}) < h[f(u,x_{2n-1}),\ f(u,T_1u),\ f(x_{2n-1},T_2x_{2n-1})]$$

$$< \max[f(u,x_{2n-1}),\ f(x_{2n-1},x_{2n})] \qquad \text{— (1)}$$

Similarly, we can have $f(u,x_{2n-1}) < \max[f(u,x_{2n-2}),\ f(x_{2n-1},x_{2n})]$ — (2)

We now choose $n > n_k$ and by repeated application of (1) and (2) we have $f(u,x_{2n}) < \max[f(u,x_{2n-3}),\ f(x_{2n-3},x_{2n-2})] <$ ....

$$< \max[f(u,x_{2n_k}),\ f(x_{2n_k}\ x_{2n_k+1})] < \partial$$

$$\Rightarrow \lim_{n\to\infty} f(u,x_{2n}) = 0 \Rightarrow f(u,\lim_{n\to\infty} x_{2n}) \Rightarrow 0 \Rightarrow \lim_{n\to\infty} x_{2n} = u.$$

Thus the sequence of iterate $\{(T_2T_1)^n x_0\}$ converges to u. Similarly for the case of the sequence of iterate $\{T_1\ (T_2T_1)^n x_0\}$.

Hence this completes the proof of the theorem.

2. Fixed Point Theorems in Pseudocompact Tychonoff Space

In this section our main interest is to study some theorems on fixed point in pseudocompact Tychonoff space using rational expression. We first give the following

Definition-[5] :- A topological space is said to be pseudocompact if every real valued continuous function on it is bounded.

It is known [5] that — every compact sapce is pseudocompact but the converse is not true; and the product of two Tychonoff spaces is Tychonoff but the product of two pseudocompact Tychonoff spaces may not be pseudocompact Tychonoff.

We now formulate and prove our main theorems:

Theorem-2.1 :- Let X be a pseudocompact Tychonoff space and $T: X \to X$ be continuous. Suppose that $f: X \times X \to \mathbb{R}^+$ is a continuous function such that

i] $f(x,y) = 0$ if and only if $x = y$ for every $x,y \in X$;

ii] $f(x,y) = f(y,x)$ for every $x,y \in X$.

If T satisfies the inequality

$$*]\ f(Tx,Ty) < \frac{\max[f(x,y)\, f(x,Ty),\ f(x,y)\, f(y,Tx),\ f(x,Tx)\, f(x,Ty), f(y,Ty)\, f(y,Tx)]}{\max[f(x,Ty),\ f(y,Tx)]}$$

for every pair $x,y \in X$, provided that the denomiator does not vanish;

otherwise $f(Tx, Ty) = 0$. Then T has a fixed point.

Proof : When the denominator of *] vanishes for some pair $x,y \in X$ then T has fixed point.

Now suppose that the denominator of *] does not vanish. We defing a mapping ϕ : $X \longrightarrow \mathbb{R}^+$ by $\phi(x) = f(Tx,x)$ $\forall x \in X$.

Clearly ϕ is continuous. Since X is pseudocompact every real valued continuous function on X is bounded and attains its bounds. Hence there exists $z \in X$ such that $\phi(z) = \inf [\phi(x) / x \in X]$ — (1)

We shall show that z is a fixed point of T. Suppose not. Then we have $\phi(Tz) = f(T^2z,Tz)$

$$< \frac{\max [f(Tz,z)\, f(Tz,Tz),\ f(Tz,z)\, f(z,T^2z),\ f(Tz,T^2z)\, f(Tz,Tz),\ f(z,Tz)\, f(z,T^2z)]}{\max [f(Tz,Tz), f(z,Tz)]}$$

$= f(Tz,z) = \phi(z)$ — which is impossible. Thus z must be a fixed point of T. This completes the proof of the theorem.

It is well known that the notion of "compactness" and "pseudocompactness" coincide in a metric space. Threrefore, just as a particular case we can have the following

Corollary-2.1.1 :- Let(X,d) be a compact metric space and $T:X \longrightarrow X$ satisfy condition-*] of theorem-2.1 with $f(x,y) = d(x,y)$ for all $x,y \in X$, $x \neq y$. Then T has a fixed point in X.

In the next theorem we shall study the common fixed point of two self mappings defined on X, a pseudocompact Tychonoff space.

Theorem-2.2 :- If $S,T:X \longrightarrow X$ are continuous and satisfy the inequality

$$+]\ f(Sx,Ty) < \frac{\max [f(x,y)\, f(x,Ty),\ f(x,y)\, f(y,Sx),\ f(x,Sx)\, f(x,Ty), f(y,Ty)\, f(y,Sx)}{\max [f(x,Ty), f(y,Sx)]}$$

for all $x,y \in X$, $x \neq y$ where f is as in theorem-2.1 with $Sx \neq Ty$, provided that the denominator does not vanish for any $x,y \in X$: otherwise $f(Sx,Ty) = 0$.

Then S and T have a common fixed point in X.

Proof : The theorem is obvious when the denominator of +] vanishes for some pair $x,y \in X$.

Now suppose that the denominator of +] does not vanish. Since S,T and f are continuous in their resprctive domains and X is pseudocompact Tychnoff, hence there exist $z,w \in X$ such that

$f(z,Sz) = \inf [f(x,Sx) \mid x \in X]$ and

$f(w,Tw) = \inf [f(x,Tx) \mid x \in X]$.

Assuming that $Sz \neq TSz$, we have

$f(Sz,TSz)$

$$< \frac{\max [f(z,Sz)\, f(z,TSz),\, f(z,Sz)\, f(Sz,Sz),\, f(z,Sz)\, f(z,TSz),\, f(Sz,TSz)\, f(Sz,Sz)]}{\max [f(z,TSz),\, f(Sz,Sz)]}$$

$= f(z,Sz)$

$\Rightarrow f(w,Tw) < f(z,Sz)$.

Similarly, we can show that $f(z,Sz) < f(w,Tw)$ on assuming that $Tw \neq STw$.

So we must have either $Sz = TSz$ or $Tw = STw$.

Suppose that $Sz = TSz$ so that $t = Sz$ is a fixed point of T. We shall show that t is also a fixed point of S. Assuming that $St \neq t = Tt$, then $f(St,t) = f(St,Tt)$

$$< \frac{\max[f(t,t)\, f(t,Tt),\, f(t,t)\, f(t,St),\, f(t,St)\, f(t,Tt),\, f(t,Tt)\, f(t,St)]}{\max [f(t,Tt),\, f(t,St)]}$$

$= 0$ ———— which is absurd.

Hence t is a common fixed point of both S and T.

Ohter possibility that $Tw = STw$ will also lead to the same conclusion.

This completes the proof of the theorem.

3. Fixed Point Theorems in Locally Convex Hausdorff Linear Topological Space :

This section is devoted to the study of fixed point theorems of an operator on a weakly compact convex subset of a locally convex Hausdorff lineear topological space (LCHLT space) with some restriction on the operator. Throughout this section we shall use '$\rightarrow$' and '$\rightharpoonup$' to denote respectively the strong and weak convergences of a sequence. Before going to the rigour of the theorems we need some definitions and results, stated as follows:

Definition-3.1[2] :- Suppose that X and Y are two topological spaces. A mapping $T:X \rightarrow X$ is said to be demiclosed if for any sequence $\{x_n\}_n \subset X$, $x_n \rightharpoonup x_0$, $Tx_n \rightarrow Y$ would imply $Tx_0 = y$.

Let X be a LCHLT space with a countable family of continuous seminorms $P = \{p_\alpha : \alpha \in I\}$ which induces a topology on X.

Definition-3.2[7] :- Let $K \subset X$. A mapping $T:X \rightarrow X$ is said to be asymptotically non-expansive on K if there is a sequence $\{k_i\}_i$ of real numbers such that $k_i \rightarrow 1$ as $i \rightarrow$

∞ and for which

$[\forall x, y \in X]$ $[\forall \alpha \in I]$, $p_\alpha(T^i x - T^i y) \leq k_i p_\alpha(x-y)$.

It is obvious that for any asymptotically non-expansive mappings it may be a assumed that $k_i \geq 1$ and that $k_{i+1} \leq k_i$ for $i \in$ IN, so henceforth we shall assume this to be the case

Definition-3.3[1] :- A family $F = \{f_x : x \in K\}$ of functions from $[0,1]$ into a subset K of a linear topological space X will be called a jointly weakly continuous if $t \to t_o$ in $[0,1]$ and $x \to x_o$ in K then $f_x(t) \to f_{x_o}(t_o)$ in K.

Theorem-b[6] :- Every weakly compact subset of a locally convex space is complete.

Theorem-c[3] :- Every weakly closed convex subset of a weakly compact set in a locally convex space is seqentially compact.

We shall now turn our attention to prove our main theorems.

Theorem-3.1 :

Let K be a weakly compact convex subset of a locally convex Haudorff linear topological space X. If $T:K \longrightarrow K$ is nonexpansive and $(I-T)$ is demiclosed then T has a fixed point in K.

Proof. – We can assume, without any loss of generally that $o \in K$. Let $\{t_n\}_n$ be a sequence of numbers with $0 < t_n < 1$ such that $t_n \longrightarrow 1$ as $n \longrightarrow \infty$, Consider a mapping $T_n : K \longrightarrow K$ for each n such that

$$[\forall x \in K]\ [\forall n \in \mathrm{IN}],\ T_n x = t_n Tx + (1-t_n)0 = t_n Tx \tag{1}$$

We observe that T_n is well defined and it maps K into itself. So we have $[\forall x, y \in K]$ $[\forall n \in$ IN$]$, $p_\alpha(T_n x - T_n y) = p_\alpha(t_n(Tx-Ty)) \leq t_n p_\alpha(x-y)$

$\Rightarrow$ each T_n is a contraction mapping.

Since K is weakly compact in X,K is weakly closed and hence strongly closed [as a metric space where the metric is induced by the countable family of seminorms]. Therefore Theorem-B yields that each T_n has a unique fixed point, say u_n in K i.e. for all $n \in$ IN

$$T_n u_n = u_n \tag{2}$$

Again, folloing from Theorem–C,K is weakly sequentially compact;

so there is a subsequence $\{u_{n_k}\}_k$ of the sequence $\{u_n\}_n$ such thaata $u_{n_k} \longrightarrow a$ as k

$\to \infty$ where $a \in K$.

Now using (1) and (2) as required, we have

$$\begin{aligned}(I-T)u_{n_k} &= u_{n_k} - Tu_{n_k} = u_{n_k} - t_{n_k}Tu_{n_k} + t_{n_k}Tu_{n_k} - Tu_{n_k}\\ &= u_{n_k} - T_{n_k}u_{n_k} + (t_{n_k} - 1)Tu_{n_k}\\ &= (t_{n_k} - 1)Tu_{n_k} \to 0 \text{ as } k \to \infty, \text{ as } Tu_{n_k} \text{ is bounded,}\end{aligned}$$

K being weakly compact.

Since (I–T) demiclosed, hence $(I-T)a = 0 \Rightarrow Ta = a$.

Consequently, a is a fixed point of T in K.

Hence the result.

Theorem-3.2 :-

Let K be a weakly compact convex subset of a locally convex Hausdorff linear topological space X and suppose $F = \{f_x : x \in K\}$ is a family of jointly weakly continuous functions on [0,1] into K such that

i) for each $x \in K$, $f_x(1) = x$; and

ii) for a function $\phi : (0,1) \to (0,1)$, for every x,y in K and for t in [0,1]

*] $p_\alpha(f_x(t) - f_y(t)) \leq \phi(t)p_\alpha(x-y)$.

Then any asymptotically nonexpansive mapping $T : K \to K$ has a fixed point in K.

Proof :- To prove the theorem we need the following

Lemma :- If K is a bounded subset of X and $T:K\to K$ is asymptotically nonexpansive then T will satisfy for every $x \in K$

1] $\lim_{i\to\infty} \sup \{\sup_{y\in K} [p_\alpha(T^i x - T^i y) - p_\alpha(x-y)]\} \leq 0$.

Proof Of Lemma :- The proof runs just as parallel to that of Kirk[7] replacing only the 'norm' by 'seminorm' as follows:

T is asymptotically nonexpansive

$\Rightarrow \exists$ a sequence $\{k_i\}_1$ of real numbers such that $k_i \to 1$ as $i \to \infty$ and for which we have

$[\forall x,y \in K]\ [i \geq N_o \in \mathbb{N}]\ [\forall \alpha \in I]$, $p_\alpha(T^i x - T^i y) \leq k_i p_\alpha(x-y)$.

Thus,

$[p_\alpha(T^ix-T^iy)-p_\alpha(x-y)] \le (k_i-1)p_\alpha(x-y) \le |k_i-1|\partial(K)$ where $\partial(K)$ is the diameter of K

$\Rightarrow \lim_{i\to\infty} \sup\{\sup_{y\in K}[p_\alpha(T^ix-T^iy)-p_\alpha(x-y)]\} \le \lim_{i\to\infty} |k^i-1|\partial(K) = 0$ for each $x \in K$, which completes the lemma.

PROOF OF THEOREM: We define for each n = 1,2,..., $\beta_n = 1/k_n$.

Suppose that $\{T_n : n \in \mathbb{N}\}$ is a family of mappings such that

2] $[\forall x \in K]$ $[\forall n \in \mathbb{N}]$, $T_nx = f_{T^nx}(\beta_n)$

Then clearly for each $n \in \mathbb{N}$, $T_n : K \to K$.

We choose ϕ in such a way that, $[\forall n \in \mathbb{N}]$ $\phi(\beta_n)k_n < 1$.

Since T is asymptotically nonexpansive, it follows that

$$p_\alpha(T_nx-T_ny) = p_\alpha(f_{T^nx}(\beta_n)-f_{T^ny}(\beta_n)) \le \Phi(\beta_n)P_\alpha(T^nx-T^ny) \le \Phi(\beta_n)k_np_\alpha(x-y)$$

Hence for each n, T_n is a contraction mapping and therefore continuous over K.

Since the weak topology is Hausdorff and K is weakly compact, hence by Theorem-B, K is complete.

So each T_n has a unique fixed point, say u_n in K i.e. $T_nu_n = u_n$, $\forall n \in \mathbb{N}$.

Again Theorem-C yields that K is weakly sequentially compact. This enables us to find out a subsequence $\{u_{n_i}\}_i$ of the sequence $\{u_n\}_n$ such that $u_{n_i} \rightharpoonup u$ as $i \to \infty$ where $u \in K$.

So, $T_{n_i}\left(u_{n_i}\right) = u_{n_i} \rightharpoonup u$ as $i \to \infty$.

The convexity of K in a locally convex space X guarantees that

$T_{n_i}\left(u_{n_i}\right) = u_{n_i} \to u$ as $i \to \infty$.

We now choose $\varepsilon > 0$, & construct $K_x \subseteq K$ for each $x \in K$ as follows :

$$K_x = \bigcup_{k=1}^{\infty} \bigcap_{r=k}^{\infty} [S(T^rx;\varepsilon) \cap K].$$

Therefore we can write $K = \bigcup_{x \in K} K_x$. Thus we have

$u \in K \Rightarrow Tu \in K \Rightarrow [\exists\ x \in K]$, $Tu \in K_x \Rightarrow [\exists\ k = i \in \mathbb{N}]\ [x \in K]$, $Tu \in \bigcap_{r=i}^{\infty} [S(T^r x; \varepsilon) \cap K]$

$$\Rightarrow [Vr \geq i]\ p_\alpha(T^r x - Tu) < \varepsilon,$$

i.e. $T^n x \longrightarrow Tu$ as $n \longrightarrow \infty$.

Let $\partial > 0$ and using 1] in the Lemma choose $M > 0$ so that $n_i \geq M$

$\Rightarrow \sup_{y \in K} [p_\alpha\left(T^{n_i}(Tu) - T^{n_i}(y)\right) - p_\alpha(Tu - y)] \leq 1/3\ \eta$.

Now, given $n_i \geq M$, since $T^n x \longrightarrow Tu$, so $\exists\ m > n_i$ such that

$p_\alpha(T^m x - Tu) \leq 1/3\ \eta$ and $p_\alpha\left(T^{m-n_i}(x) - Tu\right) \leq 1/3\ \eta$.

Thus $n_i \geq M \Rightarrow p_\alpha\left(T^{n_i+1}(u) - Tu\right) \leq p_\alpha(Tu - T^m x) + p_\alpha\left(T^m x - T^{n_i}(Tu)\right)$

$$= p_\alpha(Tu - T^m x) + \left[p_\alpha\left(T^{n_i}(Tu) - T^{n_i}(T^{m-n_i}(x))\right) - p_\alpha\left(Tu - T^{m-n_i}(x)\right) + p_\alpha\left(Tu - T^{m-n_i}(x)\right)\right.$$

$$\leq p_\alpha(Tu - T^m x) + \sup_{y \in K}\left[p_\alpha\left(T^{n_i}(Tu) - T^{n_i}(y)\right)\right.$$

$$p_\alpha(Tu - y) + p_\alpha\left(Tu - T^{m-n_i}(x)\right)$$

$$\leq 1/3\ \eta + 1/3\ \eta + 1/3\ \eta = \eta,$$

$\Rightarrow T^{n_i}(u) \longrightarrow Tu$ as $i \longrightarrow \infty$.

Thus we have $\beta_{n_i} \longrightarrow 1$ and $T^{n_i}(u) \longrightarrow Tu$ as $i \longrightarrow \infty$. Therefore, by definition – 3.3 and using 2], we have

$$T_{n_i} u = f_{T^{n_i}}(u)^{(\beta_{n_i})} \longrightarrow f_{Tu}(1)$$

i.e. $T_{n_i} u \longrightarrow Tu$ as $i \longrightarrow \infty$. But each T_{n_i} being continuous over K, we have $[\forall i \in \mathbb{N}]$ $T_{n_i} u = T_{n_i}(\lim_{i\to\infty} u_{n_i}) = \lim_{i\to\infty} T_{n_i}(u_{n_i}) = \lim_{i\to\infty} u_{n_i} = u$.

Since X is Hauasdorff, we must have $Tu = u$. Hence T has a fixed point in K. This completes the proof of the theorem.

References

1. W. G. DOTSON, Jr. : On fixed points of non-expansive mappings in nonconvex sets Proc.Amer.Math.Soc., 38,No.-1,(1973),155-156.

2. ----------- : Fixed point theorems for non-expansive mappings on starshapped subsets of Banach spaces, J.Lond.Math.Soc., (2),4,(1972), 408-410.

3. N. DUNFORD and J. T. SCHWARTZ. : "Linear Operators", Interscience Publishers, a division of John Wiley & Sons, INC., New York, Pt.-I, 1958, 430-433.

4. M. EDELSTEIN. : On fixed point & periodic points under contractive mappings, J Lond. Math. Soc, 37(1962), 74-79.

5. R. ENGELKING. : "Outline of General Topology", North Holand Publishig Co., Amsterdam, John Wiley & Sons, INC., New York, 1968,146-151.

6. J. L. KELLEY and I. NAMIOKA. : 'Linear Topological Spaces", Princeton, N.J., Van Norstand Publishing Co., 1964,156.

7. W. A. KIRK. : Fixed point theorems for non-Lipschitzian mappings of asymptotically non-expansive type, Isreal J.Math. vol.17, 1974, 339-346.

8. J. MATKOWSKI. : Fixed point theorems for mapping with a contractive iterate at a point, Proc.Amer.Math.Soc., 62,1977,344-348.

9. J. SCHAUDER. : Der Fixpunktsatz in Funktionalraumen, Studia Math., 2, 1930, 171-180

10. S. P. SINGH and B. A. MEADE : On common fixed point theorems, Bull.Austr.Math.Soc., 16, 1977, 49 - 53.

11. A. N. TYCHONOFF : Ein Fixpunktsatz, Math.Ann.III,1935,767-776.

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SOME CHARACTERIZATIONS OF S-CLOSED FUZZY TOPOLOGICAL SPACES

S. MALAKAR and M. N. MUKHERJEE

Abstract : The present paper is a continuation of the study of S-closed fuzzy topological spaces, introduced and investigated earlier by Mukherjee and Ghosh [7]. The main purpose here is to give some new characterizations of such a space.

KEYWORDS S-closed fuzzy topological space, θ-semiadherent point, SC-accumulation point, θ-semiadherent point of a fuzzy net.

1980 Mathematics Subject Classification Code 54A40.

Starting from the paper of Thompson [12], S-closed topological spaces have already been studied quite extensively by many authors (e.g. see [2,6,8,9]), such a study in its pursuit has involved various basic notions of topology and has yielded many useful and applicable results implicating many other topological concepts. The idea was extended to fuzzy topological spaces [3] (henceforth fts, for short) by Mukherjee and Ghosh [7]. Our endeavour here is to add a few more characterizations of such a fuzzy space with the help of the introduced concepts of θ-semi-open fuzzy sets, θ-semiadherent point of nets.

Simply by X, we shall mean an fts (X,T). Following Pu and Liu [10], a fuzzy point in X with support x(∈ X) and value α ($0 < \alpha \leq 1$) will be denoted by x_α. For two fuzzy sets A and B in X, the notation AqB will mean that A is q-coincident with B, i.e., $A(x) + B(x) > 1$ for some $x \in X$. For any fuzzy set A, we shall write clA, intA, (1-A) and suppA to denote the closure,interior, complement and support of A respectively in the fts X. The constant functions taking on respectively the values O and 1 on X will be designated by O_X and 1_X respectively. For the notions of semiopen, semiclosed, regularly open and regularly closed fuzzy sets which are defined in the same way as in the classical case, we refer to Azad [1]. A fuzzy set A in X is said to be a semi-q-nbd of a fuzzy point x_α iff there is a semiopen fuzzy set V in X with $x_\alpha \, q \, V \leq A$ [4]. The semiclosure of a fuzzy set A in X, to be denoted by sclA, is the union of all fuzzy points x_α such that every semiopen semi-q-nbd of x_α is q-coincident with A, or equivalently, sclA is the intersection of all semiclosed fuzzy sets containing A [4]. In the usual manner, one can define the semiinterior of a fuzzy set A, written sintA, by means of the relation sintA = 1 — scl(1-A). It is then easy to check that sintA is the union of all semiopen fuzzy sets contained in A.

we now recall the following definition of an S-closed fts as given by Mukherjee and Ghosh [7] :

Definition 1 An fts X is said to be S-closed iff for every cover ϑ of X (i.e., $\cup\vartheta = 1_X$) by semiopen fuzzy sets, there exists a finite subcollection ϑ_0 of ϑ such that $\cup\{\mathrm{cl}U : U\in\vartheta_0\} = 1_X$.

Although it is known that for a finite collection ϑ of fuzzy sets, $\cup\{\mathrm{scl}U : U\in\vartheta\} \leq \mathrm{scl}(\cup\vartheta) \leq \cup\{\mathrm{cl}U : U\in\vartheta\}$ (where the inclusions cannot be replaced by equality, in general), it is interesting to observe the following equivalent formulation of an S-closed fts in terms of semiclosures instead of closures.

Theorem 2 An fts X is S-closed iff for every cover ϑ of X by semiopen fuzzy sets there exists a finite subfamily ϑ_0 of ϑ such that $\mathrm{scl}(\cup\{U : U\in\vartheta_0\}) = 1_X$.

Proof If X is S-closed, then every cover ϑ of X by semiopen fuzzy sets has a finite subfamily ϑ_0 such that

$\cup\{\mathrm{cl}U : U\in\vartheta_0\} = 1_X$. Then

$0_X = \cap\{1-\mathrm{cl}U : U\in\vartheta_0\} = \cap\{\mathrm{int}(1-U) : U\in\vartheta_0\} = \mathrm{int}\cap\{(1-U) : U\in\vartheta_0\}$.

Put $\cap\{(1-U) : U\in\vartheta_0\} = V$. Since $\mathrm{int}\,V = 0_X$, V cannot contain any non-null semiopen fuzzy set. Thus

$0_X = \mathrm{sint}[\cap\{(1-U) : U\in\vartheta_0\}] = \mathrm{sint}[1-\cup\{U : U\in\vartheta_0\}]$

$= 1-\mathrm{scl}\{U : U\in\vartheta_0\}$, i.e., $\mathrm{scl}(\cup\{U : U\in\vartheta_0\}) = 1_X$.

Converse follows from the fact that for any finite collection ϑ of fuzzy sets in X,

$\mathrm{scl}(\cup\{U : U\in\vartheta\}) \leq \mathrm{cl}(\cup\{U : U\in\vartheta\}) = \cup\{\mathrm{cl}U : U\in\vartheta\}$.

We now list some characterizations of an S-closed fts as found in [7] for future reference.

Theorem 3 The following statements are equivalent for an fts X :

(a) X is S-closed.

(b) Every cover of X by regularly closed fuzzy sets has a finite subcover.

(c) Every family of regularly open fuzzy sets having the finite intersection propoerty has non-null intersection.

(d) For each family $\mathfrak{I}$ of semiclosed fuzzy sets with $\cap\mathfrak{I} = 0_X$, there exists a finite subfamily $\mathfrak{I}_0$ of $\mathfrak{I}$ such that $\cap\{\mathrm{int}F : F\in\mathfrak{I}_0\} = 0_X$.

It is now our intention to introduce a class of fuzzy sets which with regard to S-closedness will behave in the same way as is done by open fuzzy sets vis-a-vis compactness of fts's.

Definition 4 A fuzzy point x_α is to be a θ-semiadherent point of a fuzzy set A in an fts X iff for each semiopen semi-q-nbd V of x_α, clV $\not q$ A. The union of all θ-semiadherent points of A is called the θ-semiclosure of A and is denoted by $\theta\text{-cl}_s A$. A is called θ-semiclosed iff $A = \theta\text{-cl}_s A$. A fuzzy set B is called θ-semiopen iff (1-B) is θ-semiclosed.

Remark 5 For two fuzzy sets A,B in an fts X, $\theta\text{-cl}_s A \leq \theta\text{-cl}_s B$, whenever $A \leq B$.

Theorem 6 For any fuzzy set A in X,

$\theta\text{-cl}_s A = \cap\{V : A \leq V$ and V is a regularly open fuzzy set$\}$.

Proof Let $x_\alpha \leq \theta\text{-cl}_s A$. Then there exists a semiopen semi-q-nbd U of x_α such that clU $\not q$ A. Thus $A \leq 1\text{-clU}$ (=V, say). Then V is a regularly open fuzzy set in X containing A such that $x_\alpha > 1\text{-U} \geq 1\text{-clU} = V$. Thus $x_\alpha \leq$ R.H.S.

Conversely, let $x_\alpha \leq$ R.H.S. Then there exists a regularly open fuzzy set $V \geq A$ such that $x_\alpha \leq V$. Then (1-V) is a regularly closed set and hence a semiopen fuzzy set in X such that $x_\alpha q(1\text{-V})$. But $\text{cl}(1\text{-V}) = (1\text{-V})$ $\not q$ A. Hence $x_\alpha \leq \theta\text{-cl}_s A$.

Corollary 7 A regularly open (regularly closed) fuzzy set is θ-semiclosed (θ-semiopen).

Theorem 8 For any fuzzy set A in X

$\theta\text{-cl}_s\ (\theta\text{-cl}_s A) = \theta\text{-cl}_s A$.

Proof Obviously, R.H.S. $\leq$ L.H.S. Let now $x_\alpha \leq$ L.H.S. and let U be a semiopen semi-q-nbd of x_α. Then $(\text{clU})q(\theta\text{-cl}_s A)$ so that there exists $y \in$ supp (clU)$\cap$supp$(\theta\text{-cl}_s A)$ such that $(\text{clU})\ (y) + \beta > 1$, where $\beta = (\theta\text{-cl}_s A)\ (y)$. Since clU is a semi-open fuzzy set and Y_β q clU, we have cl(clU) qA i.e., clU q A. Hence $x_\alpha \leq \theta\text{-cl}_s A$.

Corollary 9 $\theta\text{-cl}_s A$ is θ-semiclosed, for any fuzzy set A in X.

Lowen [5] defined a family $\Im$ of fuzzy sets in an fts X to be a prefilterbase iff $0_X \notin \Im$, and for any $F_1, F_2 \in \Im$, there exists $F \in \Im$ such that $F \leq F_1 \cap F_2$. Now, following the line of proof of Theorem 2.10 of [7] one can easily arrive at the result :

Theorem 10 An fts X is S-closed iff for any prefilterbase $\Im$ on X, there exists a fuzzy point x_α such that $x_\alpha \leq \cap\{\theta\text{-cl}_s F : F \in \Im\}$.

In the following two theorems we characterize S-closed fts's by means of θ-semiopen and θ-semiclosed sets.

Theorem 11 An fts X is S-closed iff every family of θ-semiclosed fuzzy sets with the finite intersection property has non-null intersection.

Proof Let X be S-closed. Let $\vartheta = \{A_\lambda : \lambda \in I\}$ (where I is an index set) be a family of θ-semiclosed fuzzy sets with the finite intersection property. Let $\Im$ be the collection of all finite intersections of members of ϑ. Then $\Im$ is a prefilterbase. By Theorem 10, there is a fuzzy point x_α in X such that $x_\alpha \leq \theta\text{-cl}_s A_\lambda = A_\lambda$, for all $\lambda \in$ I.

The converse follows from Theorem 3((c) => (a)) and Corollary 7.

Theorem 12 An fts X is S-closed iff every cover of X by θ-semiopen fuzzy sets of X has a finite subcover.

Proof Let $\vartheta = \{U_\alpha : \alpha \in I\}$ be a cover of an S-closed fts X by θ-semiopen fuzzy sets of X. If ϑ has no finite subcover, then for each finite subset I_0 of I, $\bigcap_{\alpha \in I_0} (1-U_\alpha) = 1- \bigcup_{\alpha \in I_0} U_\alpha \neq 0_X$.

Thus $\{(1-U_\alpha) : \alpha \in I\}$ is a family of θ-semiclosed fuzzy sets in X with finite intersection property. By Theorem 11, $\bigcap_{\alpha \in I} (1-U_\alpha) \neq 0_X$. i.e., $\cup\{U_\alpha : \alpha \in I\} \neq 1_X$, proving that ϑ is not a cover of X, which is a contradiction. The converse follows from Theorem 3 and the fact that every regularly closed fuzzy set is θ-semiopen.

Remark 13 The necessity parts of Theorems 11 and 12 improve respectively the results "(a) => (c)" and "(a) => (b)" of Theorem 3, since a θ-semiopen (θ-semiclosed) fuzzy set may not be regularly closed (resp. regularly open) as is shown below.

Example 14 On the set R of reals, consider the fuzzy topology $T = \{C_\alpha : \frac{1}{3} < \alpha \leq \frac{2}{3}\} \cup \{0_X, 1_X\}$, where $C_\alpha(x) = \alpha$, for all $x \in R$. Consider the fuzzy set $A = C_{1/3}$. Then the family $\Im$ of all regularly open fuzzy sets containing A is given by

$\Im = \{C_\alpha : \frac{1}{3} < \alpha < \frac{2}{3}\} \cup \{1_X\}$. Clearly $\cap \Im = A$. Thus by Theorem 6, A is θ-semiclosed, but A is clearly not a regularly open fuzzy set in (R,T).

Definition 15 A fuzzy point x_α in an fts X is said to be a θ-semiadherent point of a fuzzy net $\{S_n : n \in (D, \geq)\}$ iff for each semiopen semi-q-nbd V of x_α, and each $m \in D$ there exists $n \in D$ such that $n \geq m$ and S_n q clV. In such a case we shall write $x_\alpha \leq$ θ-sad S_n.

Theorem 16 An fts X is S-closed iff every fuzzy net in X has a θ-semiadherent point.

Proof Let the fts X be S-closed and let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in X, without any θ-semiadherent point. For each fuzzy point x_α, there exists a semiopen semi-q-nbd U_{x_α} and an $n \in D$ such that $S_m \; q \; \mathrm{cl}U_{x_\alpha}$, for all $m \geq n$ ($m \in D$). Let R denote the family of all such U_{x_α}'s, where x_α's are fuzzy points in X. Consider the family $\vartheta = \{1\text{-cl}U : U \in R\}$ of regularly open fuzzy sets in X. Then ϑ has finite intersection property. In fact, given any finite subcollection $\vartheta_0 = \{1\text{-cl}U_i : i = 1,2,\ldots,k\}$ of ϑ there exists $n_i \in$

D such that $S_m \le 1\text{-cl}U_i$, for all $m \in D$ with $m \ge n_i$ $(i = 1,2,\dots,k)$. There exists $n \in D$ such that $n \ge n_1,\dots, n_k$ in D. Then for $m \ge n$ $(m \in D)$, $S_m \le \bigcap_{i=1}^{k} (1-\text{cl}U_i)$, i.e,. $\cap\, \vartheta_o \ne O_X$. Since X is S-closed, by Theorem 3 ((a) => (c)) there is a fuzzy point $y_\mu \le \cap\, \{1\text{-Cl}U : U \in R\} = 1 - \cup\{\text{cl}U : U \in R\} \le 1\text{-}U$, for all $U \in R$ ... ... (1)

But for the fuzzy point y_μ, there exists $U_{y_\mu} \in R$ such that $y_\mu \; q \; U_{y_\mu}$, i.e., $y_\mu \le 1-U_{y_\mu}$, contradicting (1).

Conversely, let ϑ be a family of θ-semiclosed fuzzy sets with the finite intersection property. Let ϑ^* denote the family consisting of all finite intersections of members of ϑ. For each $U_\alpha \in \vartheta^*$, we choose a fuzzy point $S(U_\alpha) \le U_\alpha$. Since ϑ^* is a directed set with the usual (fuzzy) set-inclusion relation, $\{S(U_\alpha) : U_\alpha \in \vartheta^*\}$ is a fuzzy net in X. Let y_p be a θ-semiadherent point of the fuzzy net, and let $U_\alpha \in \vartheta^*$. Then for any semiopen semi-q-nbd W of y_p there exists $U_\beta \in \vartheta^*$ with $U_\beta \le U_\alpha$ such that $S(U_\beta)\; q \;\text{cl}W$ and hence $U_\alpha \; q \; \text{cl}W$. This shows that $y_p \le \theta\text{-cl}_s U_\alpha$. Since $U_\alpha \in \vartheta^*$ is arbitrary, $y_p \le \cap\, \{\theta\text{-cl}_s U_\alpha : U_\alpha \in \vartheta^*\} \le \cap\, \{\theta\text{-cl}_s U : U \in \vartheta\} = \cap\, \{U : U \in \vartheta\}$. Hence by Theorem 11, X is S-closed.

Definition 17 Let $(D, \ge)$ be a directed set, and for each $n \in D$, let S_n denote a non-null semiopen (open) fuzzy set in X. Then (S_n, D) will be called a net of semiopen (resp. open) fuzzy sets in X.

A fuzzy point x_α in X is called an S-adherent point of such a net iff for each semiopen semi-q-nbd U of x_α and for each $n \in D$, there is $m \in D$ such that $m \ge n$ and $S_m \; q \; U$.

Theorem 18 An fts X is S-closed iff every net of semiopen fuzzy sets in X has an S-adherent point.

Proof Let $\{S_n : n \in (D, \ge)\}$ be a net of semiopen fuzzy sets in an S-closed space X. For each $n \in D$, let us put $F_n = \text{scl}\,(\cup\, \{S_m : m \in D \text{ and } m \ge n\})$. Then $\mathfrak{F} = \{F_n : n \in D\}$ is a family of semiclosed fuzzy sets such that for every finite subcollection $\mathfrak{F}_o$ of $\mathfrak{F}$, $\cap\, \{\text{int}\, F : F \in \mathfrak{F}_o\} \ne O_X$. Then by Theorem 3 there exists a fuzzy point $x_\alpha \le \cap\, \mathfrak{F}$. Let $m \in D$ and W be a semiopen semi-q-nbd of x_α. Since $x_\alpha \le F_m = \text{scl}\,(\cup\, \{S_k : k \in D \text{ and } k \ge m\})$, we have $W \; q \; (\cup\, \{S_k : k \in D \text{ and } k \ge m\})$. Thus there exists an $n \in D$ with $n \ge m$ such that $W \; q \; S_n$. This shows that x_α is an S-adherent point of the net $\{S_n : n \in D\}$.

We prove a stronger sufficiency that X is S-closed if every net of open fuzzy sets in X has an S-adherent point. Indeed, let $\mathfrak{F}$ be a family of semiclosed fuzzy sets in X such that for every finite subcollction $\mathfrak{F}_o$ of $\mathfrak{F}$, $\cap\, \{\text{int}\, F : F \in \mathfrak{F}_o\} \ne O_X$. Let $\mathfrak{F}^*$ denote the collection of all finite intersections of members of $\Im$. Then $\Im^*$ can be considered as a directed set under (fuzzy) set inclusion relation. For each $F \in \Im^*$, $\text{int}F \neq 0_X$ so that $\{\text{int}\, F : F \in \Im^*\}$ is a net of fuzzy open sets. By hypothesis, it has an S-adherent point x_α (say) in X. Let V be a semiopen semi-q-nbd of x_α and let $F \in \Im$. Then there exists $G \in \Im^*$ such that $G \leq F$ and $(\text{int}G)\, q V$. Then $F\, q V$. Thus $x_\alpha \leq \text{scl}F = F$. Hence $\cap\, \Im \neq 0_X$. Then by Theorem 3 it follows that X is S-closed.

Corollary 19 An fit X is S-closed iff every net of open fuzzy sets in X has an S-adherent point.

References

1. K. K. Azad, : On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, Jour. Math. Anal. Appl. 82(1981), 14-32.
2. D. E. Cameron, : Properties of S-closed spaces, Proc. Amer. Math. Soc. 72(3) (1978), 581-586.
3. C. L. Chang, : Fuzzy topological spaces, Jour. Math. Anal. Appl. 24(1968), 182-190.
4. B. Ghosh, : Semi-continuous and semi-closed mappings and semi-connectedness in fuzzy setting, Fuzzy Sets and Systems 35(1990), 345-355.
5. R. Lowen, : Convergence in fuzzy topological spaces, General Topology Appl. 10(2) (1979), 147-160.
6. Asha Mathur : a note on S-closed spaces, Proc. Amer. Math. Soc. 74(1979), 350-352.
7. M. N. Mukherjee and B. Ghosh : On fuzzy S-closed spaces and FSC sets, Bull. Malaysian Math. Soc. (2nd series) 12(1989), 1-14.
8. T. Noiri, : On S-closed spaces, Ann. Soc. Scien. Bruxelles T.19 IV (1977), 189-194.
9. T. Noiri, : Properties of S-closed spaces, Acta Math. Acad. Scien. Hungaricae 35(1980), 431-436.
10. Pao-Ming Pu and Ying Ming Liu : Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence, Jour. Math. Anal. Appl. 76(1980), 571-599.
11. Pao-Ming Pu and Ying Ming Liu : Fuzzy topology II. Product and quotient spaces, Jour. Math. Anal. Appl. 77(1980), 20-37.
12. Travis Thompson, : S-closed spaces, Proc. Amer. Math. Soc. 60(1976), 335-338.

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SOME GENERATING FUNCTIONS OF LAGUERRE POLYNOMIALS BY OPERATIONAL METHOD

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1. Extensions of Hille-Hardy formula

Operational method is sometimes very useful and elegant in deriving many properties of special functions. By operational method here we mean ordinary differential operator method which is frequently used by researchers in recent years.

In 1960, L, Carlitz [2] first found the following operational formula for the Laguerre polynomials

$$(1.1)\quad L_n^{(\alpha)}(x) = \frac{1}{n!} \mathop{\pi}\limits_{j=1}^{n} (xD - x + \alpha + j).1.$$

Later O.V. Viskov [9] has given the following operational representation for $L_n^{(\alpha)}(x)$

$$(1.2)\quad L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} e^x (xD^2 + \alpha D + D)_n e^{-x} = \frac{1}{n!} e^x \Delta^n e^{-x}\ ;\ [\Delta = -(xD^2 + \alpha D + D)]$$

Using Viskov's operator S. K. Chatterjea and B. Ghosh [1] have proved the following extension of the well-known Hille-Hardy formula

$$(1.3)\quad \sum_{n=0}^{\infty} \frac{(m+n)!}{m!\,(1+\alpha)_n} L_{m+n}^{(\alpha)}(x)\, L_n^{(\alpha)}(y)\, t^n$$

$$= \frac{(1-t)^{-\alpha-1}}{m!} \exp \frac{x-t\,(x+y)}{1-t}\, \Delta^m \left[-\frac{x}{e^{1-t}}\ {}_0F_1\left(-;\ 1+\alpha\ ;\ \frac{xyt}{(1-t)^2}\right) \right],$$

where $\Delta = -(xD^2 + \alpha D + D)$ and $D = \frac{d}{dx}\cdot$

Noticing the above result of Chatterjea and Ghosh, A. K. Chongdar [3] has tried to obtain further extension by making use of the same technique adopted by Chatterjea and Ghosh, but unfortunately his result is not correct. In this paper we have tried to find a correct version of Chongdar's result as a consequence of our main result. Recently S. Ghosh [6] has tried to obtain a further extension of the result of Chatterjea and Ghosh. His result is as follows

$$(1.4)\quad \sum_{n=0}^{\infty} \frac{(m+n)!}{(1+\beta)_n} L_{m+n}^{(\alpha)}(x)\, L_n^{(\beta)}(y) t^n$$

$$= (1-t)^{-\alpha-1} e^x \Delta^m e^{\frac{-x}{1-t}} \sum_{k=0}^{\infty} \frac{\left(\frac{-yt}{1-t}\right)^k}{(1+\beta)_k} L_k^{(\alpha)} \frac{x}{1-t}$$

where Δ is given in (1.3).

Noticing the generating series of Ghosh as well as the generating series of A. K. Chongdar we are inclined to consider the generating series

$$\sum_{n=0}^{\infty} \frac{(m+1)_n\, (r+1)_n}{n!\, (1+\beta)_n} L_{m+n}^{(\alpha)}(x)\, L_{r+n}^{(\beta)}(y) t^n$$

Our main result in this connection can be stated in the following form

$$(1.5)\quad \sum_{n=0}^{\infty} \frac{(m+1)_n\, (r+1)_n}{n!\, (1+\beta)_n} L_{m+n}^{(\alpha)}(x)\, L_{r+n}^{(\beta)}(y) t^n$$

$$= \frac{(1-t)^{-\alpha-1}}{m!\, r!} e^{x+y} \Delta_x^m \Delta_y^r \left\{ \exp\left[-\frac{x+y(1-t)}{1-t}\right] \sum_{k=0}^{\infty} \frac{\left(\frac{yt}{t-1}\right)^k}{(1+\beta)} L_k^{(\alpha)} \frac{x}{1-t} \right\}, \quad \text{where}$$

$$\Delta_x = -\left[x D_x^2 + (\alpha+1) D_x\right], \quad D_x = \frac{d}{dx},$$

$$\text{and}\, \Delta_y = -\left[y D_y^2 + (\beta+1) D_y\right], \quad D_y = \frac{d}{dy},$$

which can be considered as a nice extension of Hille - Hardy formula and which may be compared with the result (7) cf. [8, p. 240], where it may be remarked that the result (7) contains three summations in the right hand member, whereas our result contains a single summation in the right hand member and moreover our method is based on Viskov's operational formula, which is entirely different from that used in proving (7) of [8].

Now to prove our main result, viz. (1.5) we shall require the relation (1.2) and the following results

$$(1.6)\quad L_n^{(\beta)}(y) = \sum_{k=0}^{n} \frac{(1+\beta)_n (-y)^k}{k!\,(n-k)!\,(1+\beta)_k},$$

which is the well-known explicit representation of Laguerre polynomials,

$$(1.7)\quad e^t\left[xD_x^2 + (\alpha+1)D_x\right]_{e^{-kx}} = (1-kt)^{-\alpha-1}\, e^{\frac{kx}{kt-1}},$$

which is derived by B. Ghosh [5].

We notice that

$$(1.8)\quad \sum_{n=0}^{\infty} \frac{(m+1)_n\,(r+1)_n}{n!\,(1+\beta)_n} L_{m+n}^{(\alpha)}(x)\, L_{r+n}^{(\beta)}(y) t^n$$

$$= \sum_{n=0}^{\infty} \frac{(m+1)_n\,(r+1)_n\, t^n e^{x+y}}{(1+\beta)_n\, n!\,(m+n)!\,(r+n)!} \Delta_x^{m+n} e^{-x}\, \Delta_y^{n+r} e^{-y}$$

where $\Delta_x = -\left[xD_x^2 + (\alpha+1)D_x\right]$, $D_x = \frac{d}{dx}$,

and$\Delta_y = -\left[yD_y^2 + (\beta+1)D_y\right]$, $D_y = \frac{d}{dy}$,

$$= \sum_{n=0}^{\infty} \frac{t^n\, e^{x+y}}{m!\,n!\,r!\,(1+\beta)_n} \Delta_x^m\, \Delta_y^r\, \Delta_x^n e^{-x}\, \Delta_y^n e^{-y}$$

$$= \frac{e^{x+y}}{m!\,r!} \Delta_x^m \Delta_y^r e^{-y} \sum_{n=o}^{\infty} \frac{t^n}{(1+\beta)_n} L_n^{(\beta)}(y)\, \Delta_x^n e^{-x}$$

$$= \frac{e^{x+y}}{m!\,r!} \Delta_x^m \Delta_y^r e^{-y} \sum_{n=o}^{\infty} \sum_{k=o}^{n} \frac{t^n (-y)^k}{(1+\beta)_k\, k!\,(n-k)!} \Delta_x^n e^{-x}$$

$$= \frac{e^{x+y}}{m!\,r!} \Delta_x^m \Delta_y^r e^{-y} \sum_{n=o}^{\infty} \sum_{k=o}^{\infty_o} \frac{t^{n+k} (-y)^k}{(1+\beta)_k\, k!\, n!} \Delta_x^{n+k} e^{-x}$$

$$= \frac{e^{x+y}}{m!\,r!} \Delta_x^m \Delta_y^r e^{-y} \sum_{k=o}^{\infty} \frac{(-yt)^k}{(1+\beta)_k\, k!} \Delta_x^k \sum_{n=o} \frac{t^n}{n!} \Delta_x^n e^{-x}$$

$$= \frac{e^{x+y}}{m!\,r!} \Delta_x^m \Delta_y^r e^{-y} \sum_{k=o}^{\infty} \frac{(-yt)^k}{(1+\beta)_k\, k!} \Delta_x^k e^{t\Delta_x} e^{-x}$$

$$= \frac{e^{x+y}(1-t)^{-x-1}}{m\,!\,r\,!}\,\Delta_x^m\,\Delta_y^r\,e^{-y}\sum_{n=o}^{\infty}\frac{(-yt)^k}{(1+\beta)_k\,k\,!}\,\Delta_x^k\,e^{\frac{x}{t-1}}$$

Now substitute x = (l-t)z .

$$\therefore\ D_x = \frac{1}{1-t}D_z\,,\ D_x^2 = \frac{1}{(l-t)^2}D^{2_z}$$

and $\Delta_x = -(xD_x^2 + \alpha D_x + D_x) = \frac{1}{l-t}(zD_z^2 + \alpha D_z + D_z) = \frac{\Delta_z}{l-t}$.

Therefore the right hand member of (1.8)

$$= \frac{(l-t)^{-\alpha-m-l}}{m\,!\,r\,!}\ \exp[\,y + (l-t)z]\ \Delta_z^m\ \Delta_y^r\left[e^{-y}\sum_{k=0}^{\infty}\frac{\left(\frac{yt}{t-l}\right)^k}{(1+\beta)_k\,k\,!}\,\Delta_z^k\,e^{-z}\right]$$

$$= \frac{(l-t)^{-\alpha-m-l}}{m\,!\,r\,!}\ \exp[\,y + (l-t)z]\ \Delta_z^m\ \Delta_y^r\left[e^{-y}\sum_{k=0}^{\infty}\frac{\left(\frac{yt}{t-l}\right)^k}{(1+\beta)_k}\,e^{-z}L_k^{(\alpha)}\,(z)\right]$$

$$= \frac{(l-t)^{-\alpha-l}}{m\,!\,r\,!}\,e^{x+y}\ \Delta_x^m\ \Delta_y^r\left[\exp\left(\frac{x+y(l-t)}{t-l}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{yt}{t-l}\right)^k}{(1+\beta)_k}\,L_k^{(\alpha)}\left(\frac{x}{1-t}\right)\right],$$

which completes the proof of (1.5).

It may be of interest to note that the right hand member of (1.5) can also be written in the following elegant form :

$$\frac{(l-t)^{-\alpha-l}}{m\,!\,r\,!}\,e^{x+y}\ \Delta_x^m\ \Delta_y^r\left[\exp\left(\frac{x+y(l-t)}{t-1}\right)\sum_{k=0}^{\infty}\ \left(\frac{yt}{t-l}\right)^k\left[(l+\beta)_k\right]^{-1}\sum_{l=0}^{k}\frac{(\alpha-\beta)_1}{l\,!}\ L_{k-t}^{(\beta)}\left(\frac{x}{l-t}\right)\right]$$

$$\frac{(l-t)^{-\alpha-l}}{m\,!\,r\,!}\,e^{x+y}\ \Delta_x^m\ \Delta_y^r\left[\exp\left(\frac{x+y(l-t)}{t-1}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{yt}{t-1}\right)^k}{(1+\beta)_k}\,L_k^{(\beta)}\ \left(\frac{x}{1-t}\right){}_1F_1\,\left(\alpha-\beta;1+\beta+k;-\frac{yt}{1-t}\right)\right].$$

Application of our main result

Case 1. When β = α, then we have

$$(1.9)\quad \sum_{n=0}^{\infty} \frac{(m+1)_n\ (r+1)_n}{n\,!\,(1+\alpha)_n} L^{(\alpha)}_{m+n}(x)\, L^{(\alpha)}_{r+n}(y) t^n$$

$$= \frac{(1-t)^{-\alpha-1}}{m\,!\,r\,!}\, e^{x+y}\, \Delta_x^m\, \Delta'^{\,r}_y \left[\exp\left(\frac{x+y}{t-1}\right) {}_0F_1\left(-;1+\alpha;\frac{xyt}{(1-t)^2}\right)\right],$$

where $\Delta'_y = -\left(yD_y^2 + \alpha D_y + D_y\right)$,

which is the corrected form of Chongdar's result [3].

Case 2. when $\beta=\alpha$ and $r=0$, then we have

$$(1.10)\quad \sum_{n=0}^{\infty} \frac{(m+1)_n}{(1+\alpha)_n} L^{(\alpha)}_{m+n}(x)\, L^{(\alpha)}_n(y)\, t^n$$

$$= \frac{(1-t)^{-\alpha-1}}{m\,!} \exp\left(\frac{t(x+y)-x}{t-1}\right) \Delta_x^m\, e^{\frac{x}{t-1}}\, {}_0F_1\left[-;1+\alpha;\frac{xyt}{(1-t)^2}\right]$$

which is due to Chatterjea and Ghosh [1].

Case 3. when $r = 0$, then we have

$$(1.11)\quad \sum_{n=0}^{\infty} \frac{(m+1)_n}{(1+\beta)_n} L^{(\alpha)}_{m+n}(x)\, L^{(\beta)}_n(y)\, t^n$$

$$= \frac{(1-t)^{-\alpha-1}}{m\,!}\, e^x\, \Delta_x^m\, e^{\frac{x}{t-1}} \sum_{k=0}^{\infty} \frac{\left(\frac{yt}{t-1}\right)^k}{(1+\beta)_k} L_k^{(\alpha)}\left(\frac{x}{1-t}\right),$$

which is due to S. Ghosh [5].

Case 4. when $\beta = \alpha$ and $m=r=0$, then we have

$$(1.12)\quad \sum_{n=0}^{\infty} \frac{n\,!}{(1+\alpha)_n} L^{(\alpha)}_n(x)\, L^{(\alpha)}_n(y)\, t^n$$

$$= (1-t)^{-\alpha-1} \exp\left(\frac{t(x+y)}{t-1}\right) {}_0F_1\left[-;1+\alpha;\frac{xyt}{(1-t)^2}\right]$$

which is the well-known Hille - Hardy formula.

2. General theorem on trilateral generating relation

Next we return to another work of A. K. Chongdar [4], where he derived the following Theorem on mixed trilateral generating relation for Laguerre polynomials from the view - point of Lie - group.

Theorem : we have

$$(2.1)\quad (1-w)^{-1-\alpha-m}\exp\left(\frac{-xw}{1-w}\right) G\left(\frac{x}{1-w},\, u,\, \frac{wz}{1-w}\right) = \sum_{n=0}^{\infty} w^n\, f_{r,n}(z,u)\, L_{m+n}^{(\alpha)}(x)\,,$$

$$\text{where}\quad f_{r,n}(z,u) = \sum_{k=0}^{n} \binom{m+n}{m+k} a_k\, p_{r+k}(u)\, z^k,$$

$$\text{whenever}\quad G(x,u,w) = \sum_{n=0}^{\infty} a_n\, w^n\, L_{m+n}^{(\alpha)}(x)\, P_{r+n}(u)\,,$$

for arbitraty a_n and for an arbitrary polynomial $p_n(u)$ of degree n.

It may be noted that we have changed the notation from $f_n(z,u)$ used by Chongdar to $f_{r,n}(z,u)$ as the series for $f_{r,n}(z,u)$ involves both r and n.

We like to present as alternative and elementary proof of the above relation (2.1) by operational method.

Indeed, we have

$$\sum_{n=0}^{\infty} w^n\, f_{r,n}(z,u)\, L_{m+n}^{(\alpha)}(x)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{m+n}{m+k} a_k\, p_{r+k}(u)\, L_{m+n}^{(\alpha)}(x)\, z^k\, w^n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{m+n}{m+n-k} a_{n-k}\, p_{r+n-k}(u)\, L_{m+n}^{(\alpha)}(x)\, z^{n-k}\, w^n$$

$$= \sum_{n,k=0}^{\infty} \binom{m+n+k}{k} a_n\, p_{r+n}(u)\, L_{m+n+k}^{(\alpha)}(x)\, (wz)^n\, w^k$$

$$= \sum_{n,k=0}^{\infty} \frac{a_n\, e^x\, w^k}{k\,!\,(m+n)\,!} (wz)^n\, p_{r+n}(u)\, \Delta_x^{m+n+k}\, e^{-x} \qquad \left[\Delta_x = -(xD_x^2 + \alpha D_x + D_x)\right]$$

$$= \sum_{n=0}^{\infty} \frac{a_n\, e^x\, (zw)^n}{(m+n)\,!}\, p_{r+n}(u)\, \Delta_x^{m+n} \sum_{k=0}^{\infty} \frac{1}{k!}\, w^k\, \Delta_x^k\, e^{-x}$$

$$= e^{x} \sum_{n=0}^{\infty} \frac{a_n (zw)^n}{(m+n)\,!}\, p_{r+n}(u)\, \Delta_x^{m+n} \left(e^{w\Delta x}\, e^{-x}\right)$$

$$= e^{x} \sum_{n=0}^{\infty} \frac{a_n (zw)^n}{(m+n)\,!}\, p_{r+n}(u)\, \Delta_x^{m+n} (1-w)^{-\alpha-1}\, e^{\frac{x}{w-1}}$$

$$= (1-w)^{-\alpha-1}\, e^{x} \sum_{n=0}^{\infty} \frac{a_n (zw)^n}{(m+n)\,!}\, p_{r+n}(u)\, \Delta_x^{m+n}\, e^{\frac{x}{w-1}}$$

Now put $x = (1-w)y$

$$= (1-w)^{-\alpha-m-1}\, e^{(1-w)y} \sum_{n=0}^{\infty} \frac{a_n \left(\frac{zw}{1-w}\right)^n}{(m+n)\,!}\, p_{r+n}(u)\, \Delta_y^{m+n}\, e^{-y}$$

$$= (1-w)^{-\alpha-m-1}\, e^{(1-w)y} \sum_{n=0}^{\infty} a_n \left(\frac{zw}{1-w}\right)^n p_{r+n}(u)\, e^{-y}\, L_{m+n}^{(\alpha)}(y)$$

$$= (1-w)^{-\alpha-m-1}\, e^{-wy}\, G\left(y, u, \frac{wz}{1-w}\right)$$

$$= (1-w)^{-\alpha-m-1} \exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w}, u, -\frac{wz}{1-w}\right) \quad \left[\because y = -\frac{x}{1-w}\right],$$

where $G(x,u,w) = \sum_{n=0}^{\infty} a_n\, w^n\, P_{r+n}(u)\, \dot{L}_{m+n}^{(\alpha)}(x)$

This completes our proof.

As a nice application of the theorem of Chongdar,

we put $a_n = \frac{(m+1)_n\,(r+1)_n}{n\,!\,(1+\beta)_n}$ and $P_{r+n}(u) = L_{r+n}^{(\beta)}(u)$, then from our generating relation (1.5) we have

$$G(x,u,w) = \frac{(1-w)^{-\alpha-1}}{m!r!}\, e^{x+u}\, \Delta_x^m\, \Delta_u^r \left[\exp \frac{x+u(1-w)}{w-1} \Big/ \sum_{k=0}^{\infty} \frac{\left(\frac{uw}{w-1}\right)^k}{(1+\beta)_k}\, L_k^{(\alpha)}\left(\frac{x}{1-w}\right)\right],$$

where $\Delta_u = -\left(uD_u^2 + \beta D_u + D_u\right) \& D_u = \frac{d}{du}$.

It follows therefore from (2.1) that

$$\frac{(1-w-wz)^{-\alpha-1}}{m\,!\,r\,!}\; e^{x+u}\,\Delta_x^m\,\Delta_u^r \left[\exp\left\{\frac{x+u(1-w-wz)}{wz+w-1}\right\} \sum_{k=0}^{\infty} \frac{\left(\frac{uwz}{wz+w-1}\right)^k}{(1-\beta)_k}\, L_k^{(\alpha)}\left(\frac{x}{1-w-wz}\right)\right]$$

$$(2.2)\quad = \sum_{n=0}^{\infty} w^n\, f_{r,n}(z,u)\, L_{m+n}^{(\alpha)}(x)\,,$$

where $f_{r,n}(z,u) = \sum_{k=0}^{n} \frac{(m+1)_k\,(r+1)_k}{k\,!\,(1+\beta)_k} \binom{m+n}{m+k} L_{r+k}^{(\beta)}(u)\, z^k.$

As a consequence of our result (2.2), if we put $\beta = \alpha$, then we obtain the following result

$$(2.3)\quad \frac{(1-w-wz)^{-\alpha-1}}{m\,!\,r\,!}\; e^{x+u}\,\Delta_x^m\,\Delta'^{\,r}_u \left[\exp\left(\frac{x+u(1-w)}{wz+w-1}\right) {}_0F_1\left(-,\, 1+\alpha.\, \frac{xzuw}{(wz+w-1)^2}\right)\right]$$

$$= \sum_{n=0}^{\infty} \binom{m+n}{n} w^n\, L_{m+n}^{(\alpha)}(x) \sum_{k=0}^{n} \binom{n}{k} \frac{(r+1)_k\, L_{r+k}^{(\alpha)}(u)\, z^k}{(1+\alpha)_k}\,.$$

where $\Delta_u' = -\left(uD_u^2 + \alpha D_u + D_u\right)$.

which is a correct version of Chongdar's result [4, (2.16)].

It may be worthy of mention that such an application made by Chongdar leads to incorrect result, on account of the fact that the result of Chongdar corresponding to our result (1.5) is also faulty.

Again on putting $r = O$ in (2.3) we have

$$\frac{(1-w-wz)^{-\alpha-1}}{m\,!}\; \exp\left(\frac{(x+u,wz + x(w-1)}{wz+w-1}\right) \Delta_x^m \left[\exp\left(\frac{x}{wz+w-1}\right) {}_0F_1e\left(-;\, 1+\alpha;\, \frac{xzuw}{(wz+w-1)^2}\right)\right]$$

$$(2.4)\quad = \sum_{n=0}^{\infty} \binom{m+n}{n} w^n\, L_{m+n}^{(\alpha)}(x) \sum_{k=0}^{n} \frac{(-n)_k\,(-z)^k}{(1+\alpha)_k}\, L_k^{(\alpha)}(u)\,,$$

$$= \sum_{n=0}^{\infty} \binom{m+n}{n} [w(1+z)]^n\, {}_1F_1\left(-n;\, 1+\alpha\,;\, \frac{zu}{1+z}\right) L_{m+n}^{(\alpha)}(x)\,,$$

Also on putting $m = O$ in (2.4), we have

$$(1-w-wz)^{-\alpha-1} \exp\left\{\frac{(x+u)z+x}{wz+w-1}\right\} {}_0F_1\left(-?\,;\,1+\alpha\,;\frac{x\,z\,u\,w}{(wz+w-1)^2}\right)$$

$$= \sum_{n=0}^{\infty} [w(1+z)]^{n} {}_1F_1\left(-n;\,1+\alpha\,;\frac{zu}{1+z}\right) L_n^{(\alpha)}(x)\,,$$

which may easily be compared with the known result due to E. D. Rainville [7]

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References :

1. Chatterjea, S. K. & Ghosh, B. : A remark on 'An extension of Hardy - Hille formula, Pure Math. Manuscript vol.4(1985), 227.
2. Carlitz, L. : A note on the Laguerre polynomials, Michigan Math. 7(1960), 219-228.
3. Chongdar, A. K. : A general form of extension of Hardy - Hille formula, Bull. Cal. Math. Soc. 7(1986), 232-233.
4. Chongdar, A. K. : On Generating functions of Laguerre polynomials, Bull. Cal. Math. Soc. 79(1987), 48-56.
5. Ghosh, B. : Operational derivation of some generating functions for the Laguerre polynomials, Jour. Pure Math. Vol.3(1983), 43-47.
6. Ghosh S. : Extensions of Hille - Hardy formula, Pure Math. H Manuscript Vol. 5(1986), 93-95.
7. Rainville, E. D. : Special functions, New York (1960)
8. Srivastava, H. M. and Manocha, H. L. : A treatise on generating functions. Ellis Horwood Ltd (1984).
9. Viskov, O. V. : On an identity of L. B. Redei for Laguerre polynomials (in Russian) Acta Sci Math. (Szeged) 39(1977), 24-28.

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ON BITOPOLOGICAL RIGID SUBSETS

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Abstract : A class of subsets dubbed ij-rigid sets is introduced and studied in this paper. The concept is found to have a distinctive feature from that of bitopological H-sets [6], but conincides with the notion of bitopological QHC space [5] when restricted to the whole space.

Keywords : ij-rigid, ij-H-set, ij-θ-closure, ij-θ-adherence and ij-θ-convergence of filterbase.

1980 AMS Classification (1985 revision) code : 54 E 55

The bitopological version of the well-known quasi-H-closed topological space was first introduced by Mukherjee [5]. The notion restricted suitably to the subsets gave rise to the investigation of ij-H-sets in [6]. The very purpose of the present deliberation is to introduce and study a type of sets in bitopological setting, termed rigid sets. Although this class is strictly contained in that of ij-H-sets, for the whole space the notions of bitopological quasi-H-closedness and rigidity coincide.

In this paper a space X shall always mean a bitopological space (X, Q_1, Q_2) [4], and whenever i and j both will occur we assume that $i,j \in \{1,2\}$ and $i \neq j$. For a subset A in a space X, Q_i-intA and Q_i-clA will stand for the interior and closure of A in (X,Q_i) $(i=1,2)$. A point $x \in X$ is called an ij-θ-contact point of a subset A of X [3] iff any Q_i-open neighbourhood (henceforth nbd, for brevity) U of x, $(Q_j\text{-cl}U) \cap A \neq \phi$. The set of all ij-$\theta$-contact points of A is called the ij-θ-closure of A, to be denoted by ij-θ-clA; A is called ij-θ-closed iff A = ij-θ-clA. A subset A of X is said to be ij-regularly open (ij-ro-set, for short) iff $A = Q_i$ -int Q_j-clA, and complement of an ij-ro-set is called an ij-regularly closed set (ij-rc-set, for short) [8]. Given a space (X,Q_1,Q_2), the collection $B(Q_i)$ of all ij-ro-sets forms a base for a topology Q_i^s, coarser than Q_i [8]. A point $x \in X$ is called an ij-θ-adherent point of a filterbse $\mathfrak{I}$ on X [3] iff it is an ij-θ-contact point of every member of $\mathfrak{I}$. The set of all ij-θ-aherent points of $\mathfrak{I}$ is called the ij-θ-adherence of $\mathfrak{I}$, denoted by ij-θ-ad $\mathfrak{I}$. A filterbase $\mathfrak{I}$ on X is said to be ij-θ-convergent to a subset A of X, written as $\mathfrak{S} \xrightarrow{ij-\theta} A$ iff for each Q_i-open cover ϑ of A there are a finite subfamily ϑ_0 of ϑ and a member F of $\mathfrak{S}$ such that $F \subset \bigcup_{U \in \vartheta_0} Q_j\text{-cl}U$. By $\mathfrak{S} \xrightarrow{ij-\theta} x$ we mean $\mathfrak{S} \xrightarrow{ij-\theta} \{x\}$ [6]. A point $x \in X$ is called an ij-θ-adherent point [6] of a net $\{x_\alpha : \alpha \in D\}$ iff for each Q_i -open nbd U of x and each $\alpha \in D$, there exists $\beta \in D$ with $\beta \geq \alpha$ such that $x_\beta \in Q_j$ -clU.

Definition 1 A subset A of a space X is called an ij-H-set [6] iff every Q_i -open cover of A has a finite subfamily, the Q_j-closures of whose members covers A; A is called a pairwise H-set iff it is both 12-and 21-H-set. If in addition A=X, then X is called an ij-QHC (pairwise QHC) space [5].

Definition 2 A subset A of a space (X, Q_1, Q_2) is called ij-rigid iff every Q_i -open cover of A has a finite subfamily the Q_i-interior of the union of the Q_j -closures of whose members contains A; A is called pairwise rigid iff it is both 12- and 21-rigid.

Remark 3 From the above definitions it is clear that for a bitopolgical space X, the statement of it being ij-QHC is equivalent to it being ij-rigid. Also, for a subset A of X, A is ij-rigid implies that A is an ij-H-set, but the converse is not true, in general, as is seen from the example below.

Example 4 Let Y denote the well-known example of a non-compact minimal Hausdorff space (see [1] for details) given by $Y = W \cup \{(0,1), (0,-1)\}$, where $W= \left\{\left(\frac{1}{m}, \frac{1}{n}\right) : (m, n) \in N \times N\right\} \cup \left\{\left(\frac{1}{m}, -\frac{1}{n}\right) : (m, n) \in N \times N\right\} \cup \left\{\left(\frac{1}{m}, 0\right) : m \in N\right\}$ (N being the set of natural numbers).

Let Q_1 be the topology on Y such that Q_1/W = the usual subspace topology from the plane, and a basic open set containing (0.1) (resp. (0,-1)) is of the form U_r (0,1) (resp. U_r (0, -1)) given as follows, where r = 1,2,...

$$U_r(0,1) = \{(0,1)\} \cup \left\{\left(\frac{1}{m}, \frac{1}{n}\right) : m \geq r, n \in N\right\}$$

$$U_r(0,-1) = \{(0,-1)\} \cup \left\{\left(\frac{1}{m}, -\frac{1}{n}\right) : m \geq r, n \in N\right\}.$$

Let $Q_1 = Q_2$. Let X be the subspace of (Y, Q_1, Q_2) given by :

$$X = \left\{(\frac{1}{m}, \frac{1}{n}) : (m,n) \in N \times N\right\} \cup \left\{(\frac{1}{m}, 0) : m \in N\right\} \cup \{(0,1)\}.$$

Then X is a pairwise H-set but not a pairwise rigid subset of Y.

Let us now consider the topologies $Q_1{}^*$ and $Q_2{}^*$ on X defined below :

$Q_1{}^*/(X-\{(0,1)\}) = Q_2{}^*/(X-\{(0,1)\})$ = the usual topology from the plane. For $Q_1{}^*$, a base for (0,1) is given by

$$\{U_r(0,1) = \{(0,1)\} \cup \{(\frac{1}{n}, \frac{1}{2m}) : n \geq r,\ m \in N\} : r = 1,2,\ldots\},$$

and for $Q_2{}^*$ a base for (0,1) is given by

$$\{U_r(0,1) = \{(0,1)\} \cup \{(\frac{1}{n}, \frac{1}{2m-1}) : n \geq r,\ m \in N\} : r = 1,2,\ldots\}.$$

Let $A = \{(0,1)\} \cup \{(\frac{1}{n}, 0) : n \in N\}$. Then A is a pairwise H-set as well as a pairwise rigid set.

Before moving to characterize ij-rigid subsets we need some additional results.

Lemma 5 In a space (X, Q_1, Q_2), if $A \subset X$ then

(a) ij-θ-clA $= Q_i$ − clA, for $A \in Q_j$ [3],

(b) ij-θ-clA $= \cap \{Q_i$ -clV : $A \subset V \in Q_j\}$ [6].

Lemma 6 In a space (X, Q_1, Q_2) if $\mathfrak{I}$ is a filterbase on X, then ij-θ-ad $\mathfrak{I}$ = Q_i -ad$\{V \in Q_j : F \subset V$ for some $F \in \mathfrak{I}\}$.

Proof Let $x \in Q_i$ -ad$\{V \in Q_j : F \subset V$ for some $F \in \mathfrak{I}\}$.

Now $F \in \mathfrak{I} \Rightarrow x \in Q_i$ -clV for all $V \in Q_j$ such that $F \subset V \Rightarrow x \in$ ij-θ-clF (by Lemma 5(b)). Thus $x \in$ ij-θ-ad$\mathfrak{I}$. The converse is similar.

Lemma 7. [6] An ultrafilter ϑ on a space X ij-θ-converges to a point x of X iff x is an ij-θ-adherent point of ϑ.

Theorem 8 For a space (X, Q_1, Q_2), the following statements are equivalent :

(a) A is ij-rigid in X.

(b) For each Q_j -open filterbase $\mathcal{G}$ on X such that $A \cap Q_i$ -ad $\mathcal{G} = \phi$, there is $U \in G$ such that $A \cap Q_i$ -cl$U = \phi$.

(c) For each Q_j -open filterbase $\mathcal{G}$ on X satisfying $V \cap W \neq \phi$ for all $V \in \mathcal{G}$ and each Q_i-open set W containing A, also satisfies $A \cap Q_i$ -ad $\mathcal{G} \neq \phi$.

(d) Each filterbase $\mathcal{F}$ on X, satisfying $V \cap W \neq \phi$ for all Q_j -open set V such that $F \subset V$ for some $F \in \mathcal{F}$ and for all Q_i -open set W containing A, also satisfies $A \cap$ ij-θ-ad$\mathfrak{Z} \neq \phi$.

(e) Each base ϑ for an ultrafilter on X satisfying $V \cap Q_j$ -cl$W \neq \phi$ for all $V \in \vartheta$ and for all Q_i -open set W containing A, ij-θ-converges to some point in A.

(f) For each filterbase $\mathcal{F}$ on X with ij-θ-ad $\mathcal{F} \cap A = \phi$ there are $U \in Q_i$ and $F \in \mathcal{F}$ such that $A \subset U$ and Q_j -cl$U \cap F = \phi$.

(g) For each filterbase $\mathcal{F}$ on X such $A \cap$ ij-θ-ad $\mathcal{F} = \phi$ there is some $F \in \mathcal{F}$ with $A \cap$ ij-θ-cl$F = \phi$.

(h) Each net which is frequently in Q_j -cl$W$ for each Q_i -open set W containing A has an ij-θ-adherent point in A.

Proof (a) ⟹ (b) : Let $\mathcal{G}$ be a Q_j -open filterbase on X such that $A \cap Q_i$ -ad $\mathcal{G} = \phi$. For each $x \in A$, let $x \in U_x \in Q_i$ and $V_x \in \mathcal{G}$ such that $U_x \cap V_x = \emptyset$. As $\vartheta = \{ U_x : x \in A\}$ is a Q_i -open cover of A, we have $A \subset Q_i$-int Q_j-cl $(\bigcup_{r=1}^{n} U_{x_r})$ for finitely many points $x_1, \ldots, x_n \in A$. Since $(\bigcup_{r=1}^{n} U_{x_r}) \cap (\bigcap_{r=1}^{n} V_{x_r}) = \phi$, we have $[Q_i\text{-int } Q_j\text{-cl}(\bigcup_{r=1}^{n} U_{x_r})] \cap [Q_i\text{-cl}(\bigcap_{r=1}^{n} V_{x_r})] = \phi$, and so $A \cap Q_i$-cl$U = \phi$ where $U \in \mathcal{G}$ such that $U \subset (\bigcap_{r=1}^{n} V_{x_r})$.

(b) => (c) : Obvious.

(c) => (d) : Let $\mathcal{F}$ be a filterbase on X satisfying the given condition in (d) and let, if possible, $A \cap$ ij-θ-ad$\mathcal{F} = \phi$. Then by Lemma 6 we have $A \cap Q_i$-ad $\{V \in Q_j : F \subset V$ for some $F \in \mathcal{F}\} = \phi$. Thus by (c) there exist some $W \in Q_i$ with $A \subset W$ and some $U \in \{V \in Q_j : F \subset V$ for some $F \in \mathcal{F}\}$ such that $W \cap U = \phi$ which is a contradiction.

(d) => (e) : From the given condition on ϑ we see that for any Q_j-open set W with $U \subset W$ for some $U \in \vartheta$ and for any Q_i-open set V containing A, $W \cap Q_j\text{-cl}V \neq \phi$. Then $W \cap V \neq \phi$. Thus by (d), $A \cap \text{ij-}\theta\text{-ad}\,\vartheta \neq \phi$. Since ϑ is a base for an ultrafilter $\mathcal{F}$ (say) on X, it follows from Lemma 7 that $\vartheta \xrightarrow{\text{ij-}\theta} x$ for each $x \in A \cap \text{ij-}\theta\text{-ad}\vartheta$, and (e) follows.

(e) => (f) : Let $\mathcal{F}$ be a filterbase on X such that for all $F \in \mathcal{F}$ and for all $U \in Q_i$ satisfying $A \subset U$, $F \cap Q_j\text{-cl}U \neq \phi$. Let ϑ be an ultrafilter on X generated by the base $\beta = \mathcal{F} \cup \{Q_j\text{-cl}U : U \in Q_i, A \subset U\}$. Then any $B \in \beta$ and any $U \in Q_i$ with $A \subset U$ satisfy $B \cap Q_j\text{-cl}U \neq \phi$. Hence $\vartheta \xrightarrow{\text{ij-}\theta} x$ for some $x \in A$. Since $x \in A \cap \text{ij-}\theta\text{-ad}\,\mathcal{F}$, we have $A \cap \text{ij-}\theta\text{-ad}\,\mathcal{F} \neq \phi$.

(f) => (g) : If $A \subset U \in Q_i$ and $(Q_j\text{-cl}U) \cap F = \phi$, then $A \cap \text{ij-}\theta\text{-cl}F = \phi$.

(g) => (a) : Let ϑ be a Q_i-open cover of A. Let $\mathcal{F} = \{X - \bigcup_{U \in \beta} Q_j - \text{cl}U : \beta$ is a finite subfamily of $\vartheta\}$. If $\mathcal{F}$ is not a filterbase, then $X - \bigcup_{U \in \beta} Q_j - \text{cl}U = \phi$ for some finite subset β of ϑ and then $A \subset X = Q_i\text{-int}\,Q_j\text{-cl}\,(\cup\beta)$. Thus let $\mathcal{F}$ be a filterbase on X. We have $A \cap \text{ij-}\theta\text{-ad}\,\mathcal{F} = \phi$, since for each $x \in A$, if $x \in U_x \in \vartheta$, then $x \notin \text{ij-}\theta\text{-cl}\,(X - Q_j\text{-cl}U_x)$ whereas $X - Q_j\text{-cl}U_x \in \mathcal{F}$. Therefore, $A \cap \text{ij-}\theta\text{-cl}\,(X - \bigcup_{U \in \beta} Q_j - \text{cl}U) = \phi$ for some finite sub-family β of ϑ, i.e., $A \cap Q_i\text{-cl}\,(X - \bigcup_{U \in \beta} Q_i - \text{cl}U) = \phi$. Hence $A \subset Q_i\text{-int}\,Q_j\text{-cl}\,(\cup\beta)$.

(f) => (h) : Let $\{x_n\}$ be a net which is frequently in Q_j-clW for each Q_i-open set W containing A. Then the filterbase $\mathfrak{S}$ induced by $\{x_n\}$ satisfies $F \cap Q_j\text{-cl}W \neq \phi$ for each $F \in \mathfrak{S}$ and each Q_i-open set W containing A. Thus by (f), $A \cap \text{ij-}\theta\text{-ad}\,\mathfrak{S} \neq \phi$. Each $x \in A \cap \text{ij-}\theta\text{-ad}\,\mathfrak{S}$ is then an ij-θ-adherent point of $\{x_n\}$.

(h) => (f) : Let $\mathcal{F}$ be a filterbase on X such that $F \cap Q_j\text{-cl}W \neq \phi$, for all $F \in \mathfrak{S}$ and all Q_i-open nbd W of A. Thus $\mathcal{F}^* = \{F \cap Q_j\text{-cl}W : F \in \mathcal{F}, A \subset W \in Q_i\}$ is a filterbase on X, then the net induced by $\mathcal{F}^*$ is frequently in Q_j-clW for each Q_i-open set W containing A. Hence $A \cap \text{ij-}\theta\text{-ad}\,\mathcal{F} \neq \phi$.

We have already pointed out that an ij-rigid set in X is always an ij-H-set. To investigate for the converse, we need the following lemma which incidentally gives some equivalent formulations of an ij-almost regular space, defined by Singal and Arya [8] as follows.

Definition 9 A space (X, Q_1, Q_2) is called ij-almost regular iff for each $V \in Q_i$ and each $x \in V$, there exists a Q_i -open nbd U of x such that Q_j -clU $\subset$ Q_i -int Q_j -clV.

Lemma 10 For a space (X, Q_1, Q_2) the following statements are equivalent :

(a) X is ij-almost regular.

(b) For any Q_j -open set A, ij-θ-cl(ij-θ-clA) = ij-θ-clA.

(c) For any set A($\subset$ X), ij-θ-clA = Q_i^s -clA.

(d) For any set A($\subset$ X), ij-θ-cl(ij-θ-clA) = ij-θ-clA.

Proof (a) => (c) : It is always true that Q_i^s–clA$\subset$ij–θ–clA.

Let $x \in$ ij-θ-clA and U be a Q_i -open set containing x. Then by (a), there is a Q_i -open nbd V of x such that Q_j -clV$\subset Q_i$ -int Q_j -clU. Since Q_j -clV$\cap$A $\neq \phi$, we have (Q_i–int Q_j–clU)$\cap$A $\neq \phi$ and thus $x \in Q_i^s$ -clA.

(c) => (d) : ij-θ-cl (ij-θ-clA) = ij-θ-cl(Q_i^s -clA) = Q_i^s -cl (Q_i^s -clA) = Q_i^s -clA = ij-θ-clA.

(d) => (b) : Clear.

(b) = (a) : Let C be any ij-ro-set in X and $p \in$ C. Now, A = X – C is an ij-rc-set and hence A = Q_i -cl (Q_j -intA). Put B = Q_j -intA. Then by virtue of (b) and by Lemma 5(a), ij-θ-clA = ij-θ-cl(Q_i-clB) = ij-θ-cl(ij-θ-clB) = ij-θ-clB = Q_i-clB = A. Then $p \notin$ ij-θ-clA and hence there is a Q_i -open nbd G of p such that Q_j -clG $\cap$ A = ϕ, i.e., Q_j -clG $\subset$ X – A = C. Hence X is ij-almost regular.

Theorem 11 If in space (X, Q_1, Q_2) any one of the four conditions of Lemma 10 is satisfied, then an ij-H-set of X is ij-rigid.

Proof Let $\mathcal{F}$ be a filterbase in X such that A$\cap$ij-θ-ad $\mathcal{F}$ = ϕ. Then for each $a \in$ A, there is $F_a \in \mathcal{F}$ such that $a \notin$ ij-θ-clF_a. By Lemma 10(d), ij-θ-clF_a = ij-θ-cl(ij-θ-clF_a) and hence there is a Q_i -open nbd U_a of a such that Q_j -clU_a $\cap$ij-θ-clF_a = ϕ.

$\{U_a : a \in A\}$ being a Q_i -open cover of the ij-H-set A, there is a finite subset B of A such that $A \subset \bigcup_{a \in B} Q_i - \text{cl}U_a$. Let $F \in \mathcal{F}$ such that $F \subset \bigcap_{a \in B} F_a$. Now,

$(\bigcup_{a\in B} Q_j-\mathrm{cl}U_a)\cap(\bigcap_{a\in B} ij-\theta-\mathrm{cl}F_a)=\phi$ implies $A \cap$ ij-θ-clF $= \phi$, since ij-θ-clF $\subset$ ij-θ-cl $(\bigcap_{a\in B} F_a)\subset(\bigcap_{a\in B} ij-\theta-\mathrm{cl}F_a)$ Thus by Therem 8 (g) we have that A is ij-rigid.

Theorem 12 If A is ij-rigid in X and $B\subset X$ is such that $A\cap$ij-θ-clB $= \phi$, then there are $U \in Q_i$ and $V \in Q_j$ such that $A\subset U$, $B\subset V$ and $U\cap V = \phi$.

Proof If the theorem is false, then $\mathcal{F} = \{U\cap V : A\subset U \in Q_i \text{ and } B\subset V\in Q_j\}$ forms a filterbase on X. For any Q_i-open set $W\supset A$ and any $U\cap V\in \mathcal{F}$, Q_j -clW $\cap$ $(U\cap V)\supset (W\cap U)\cap V\neq \phi$ and so $A\cap$ij-θ-ad $\mathcal{F} \neq \phi$, for otherwise it will contradict Theorem 8 (f). Again, $\phi \neq A \cap [\bigcap_{F\in F} ij-\theta-\mathrm{cl}F]\subset A\cap[\bigcap ij-\theta-\mathrm{cl}V : B\subset V\in Q_j\} = A\cap[\cap\{Q_i\ -\mathrm{cl}V : B\subset V\in Q_j\}]$ $= A \cap$ ij-θ-clB, which is a contradiction.

Kelly [4] defined a space (X, Q_1, Q_2) to be pairwise Hausdorff iff for any two distinct points x,y of X, there exist a Q_i-open nbd U of x and a Q_j–open nbd V of y such that $U\cap V = \phi$. In the next theorem we show that in this definition the role of distinct points can be replaced by that of disjoint pairwise rigid subsets. Before that we have the following :

Corollary 13 : In a pairwise Hausdorff space (X, Q_1, Q_2) an ij-rigid subset is ji-θ-closed.

Proof Let $A\subset X$ be an ij-rigid subset in X and let $x\notin A$. Since in a pairwise Hausdorff space, $\{x\} =$ ij-θ-cl$\{x\}$ for all $x \in X$ [7], $A\cap\{x\} = \phi$ implies $A\cap$ij-θ-cl$\{x\} = \phi$. Now by Theorem 12, there are $U \in Q_i$ and $V\in Q_j$ such that $A\subset U$, $\{x\}\subset V$ and $U\cap V = \phi$. Then $x \notin$ ji-θ-clA, and hence A is ji-θ-closed.

Theorem 14 A space (X, Q_1, Q_2) is pairwise Hausdorff iff any two disjoint pairwise rigid subsets A, B, can be pairwise strongly separated (i.e., there exist $U\in Q_i$ and $V\in Q_j$ such that $A \subset U$, $B\subset V$ and $U\cap V = \phi$ for i, j = 1,2 and $i \neq j$).

Proof Let A and B be two disjoint pairwise rigid subsets in a pairwise Hausdorff space (X, Q_1, Q_2). We know that in a pairwise Hausdorff space X, ij-θ-cl$\{x\} = \{x\}$, for each $x \in X$ [7]. Thus by Theorem 12, for each $b \in B$ there are $U_b \in Q_i$ and $V_b \in Q_j$ such that $A\subset U_b$, $b \in V_b$ and $U_b \cap V_b = \phi$. Now $\{V_b : b \in B\}$ being a Q_j–open cover of the ji-rigid set B, there is a finite subset B_o of B such that $B\subset Q_j$-int Q_i -cl $\bigcup_{b\in B_0} V_b = W$ (say) $\in Q_j$. If $U = \bigcap_{b\in B_0} U_b$ we find that $A\subset U\in Q_i$ and $U \cap W = \phi$.

The converse is clear.

Theorem 15 Let f : $(X, Q_1, Q_2) \to (Y,P_1, P_2)$ be a map such that f : $(X,Q_i) \to (Y,P_i)$ is continuous and open for $i = 1,2$. If A is pairwise rigid in X, then so is f(A) in Y.

Proof Let f : $X \to Y$ be a map with the stated properties and $A \subset X$ be ij-rigid in X. Suppose ϑ is a P_i-open cover of f(A). The $f^{-1}(\vartheta) = \{f^{-1}(U) : U \in \vartheta\}$ is a Q_i -open cover of A. By ij-rigidity of A, there exists a finite subcollection ϑ_o of ϑ such that $A \subset Q_i - .\text{int}\, Q_j \text{-cl} [\bigcup_{U \in \vartheta_o} f^{-1}(U)]$. Now, $A \subset Q_i$ -int Q_j -cl $f^{-1}(\cup\vartheta_o) \subset Q_i$ -int $f^{-1}[P_j\text{-cl}(\cup\vartheta_o)]$ (since f : $(X, Q_j) \to (Y, P_j)$ is continuous) $\subset f^{-1}[P_i$ -int P_j -cl $(\cup\vartheta_o)]$ (f : $(X, Q_i) \to (Y, P_i)$ being open). Thus $f(A) \subset$ Pi–int Pj–cl $(\cup\vartheta_o)$ and f(A) is then ij-rigid in Y. Now putting i,j = 1,2 $(i \neq j)$ in the above arguments the proof becomes complete.

Henceforth by $(X \text{ x } Y, T_1, T_2)$ we mean the bitopological space induced by the cartesian product of (X, Q_1, Q_2) and (Y,P_1, P_2), where $T_i = Q_i \text{ x } P_j$ for i, j = 1,2 and $i \neq j$.

Theorem 16 Let (X, Q_1, Q_2) and (Y, P_1, P_2) be two bitopological spaces and let $B \subset Y$ be ji-rigid. Let $x \in X$ and $K \subset (X \text{ x } Y, T_1, T_2)$ satisfying $(\{x\} \text{ x } B) \cap$ ij-θ-clK $= \phi$. Then there exist a Q_i -open nbd V of x in X and a P_j -open set W containing B in Y such that T_j-cl $(V \text{ x } W) \cap K = \phi$.

Proof For each $y \in B$, there exist a Q_i -open nbd V_y^x of x in X and a P_j -open nbd W_y of y with $[T_j \text{-cl} (V_y^x \text{ x } W_y)] \cap K = \phi$. Since B is ji-rigid in Y, there are finitely many points $y_1, y_2, \dots, y_n \in B$ such that $B \subset P_j$ -int P_i -cl $(\bigcup_{r=1}^{n} W_{y_r}) = W$(say) $\in P_j$. Let $V = \bigcap_{r=1}^{n} V^x_{y_r}$ Then V is a Q_i-open nbd of x such that T_j -cl $(V \text{ x } W) \cap K = \phi$.

Theorem 17 Let (X, Q_1, Q_2) and (Y, P_1, P_2) be two bitopological spaces. Further let A be ij-rigid in X and B ji-rigid in Y. If $K \subset X \text{ x } Y$ satisfies $(A \text{ x } B) \cap$ ij-θ-clK $= \phi$, then there are Q_i -open set V containing A in X and P_i -open set W containing B in Y satisfying T_j -cl $(V \text{ x } W) \cap K = \phi$.

Proof If follows from Theorem 16 that for each $x \in A$, there are a Q_i -open nbd V_x of x in X and a P_j -open set W_x in Y containing B with T_j -cl $(V_x \text{ x } W_x)$

$\cap$ K = ϕ. Now A being ij-rigid, there are finitely many $x_1, ..., x_n \in$ A such that $A \subset Q_i$ -int Q_j -cl $(\bigcup_{r=1}^{n} V_{x_r}) = V$ (say) $\in Q_i$. Let $W = \bigcap_{r=1}^{n} W_{x_r} \in P_j$ then V and W have the properties as required.

Theorem 18 Let (X, Q_1, Q_2) and (Y, P_1, P_2) be two bitopological spaces and let A be an ij-rigid set in X and B a ji-rigid set in Y. Let K be a T_i -open set in X x Y and A x $B \subset K$. Then there are a Q_i -open set V containing A in X and a P_j -open set W containing B in Y satisfying T_j-cl(V x W)$\subset T_j$ -clK.

Proof Since A x B $\subset$ K $\in T_i$ we have (A x B) $\cap$ T_i -cl (X x Y $-$ T_j -clK) = ϕ which by virtue of Lemma 5(a) implies that (A x B) $\cap$ ij-θ-cl (X x Y $-$ T_j -clK) = ϕ. Now by Theorem 17 there exist a Q_i -open set V in X containing A and a P_j -open set W in Y containing B such that T_j -cl (V x W) $\cap$ (X x Y - T_j -clK) = ϕ. Consequently, T_j -cl (V x W) $\subset$ T_j -clK.

Theorem 19 Let (X, Q_1, Q_2) and (Y, P_1, P_2) be two bitopological spaces, A and B be respectively ij-rigid and ji-rigid in X and Y respectively. Then A x B is ij-rigid in (X x Y, T_1, T_2).

PROOF Let $\vartheta \times \vartheta = \{U \times V : U \in \vartheta, V \in \vartheta\}$ be a cover of A x B in (X x Y, T_1, T_2) by T_i -basic open sets, where ϑ is a Q_i -open cover of A in X and ϑ a P_j -open cover of B in Y. Since A is ij-rigid, there exists a finite subfamily ϑ_o of ϑ such that $A \subset Q_i$ -int Q_j -cl $(\cup\vartheta_o)$ and similarly B being ji-rigid, there exist ϑ a finite subfamily ϑ_o of ϑ such that $B \subset P_j$ -int P_i -cl $(\cup\vartheta_o)$. Then A x $B \subset Q_i$ -int Q_j -cl $(\cup\vartheta_o)$ x P_j -int P_i -cl $(\cup\vartheta_o)$ i.e., A $\times$ B $\subset$ T_i -int T_j -cl $[(\cup\vartheta_o) \times (\cup\vartheta_o)]$ = T_i -int T_j -cl $[\bigcup_{U \times V \in \vartheta_o \times \vartheta_o} (U \times V)]$. Thus A $\times$ B is ij-rigid in (X x Y, T_1, T_2).

Theorem 20 Let $\{X_\alpha, \zeta_1^\alpha, \zeta_2^\alpha) : \alpha \in \Lambda\}$ be a family of spaces and for each $\alpha \in \Lambda$, let A_α be a non-empty subset of X_α. A necessary and sufficient condition for πA_α to be ij-rigid in (π X_α, ζ_1, ζ_2,) is that A_α is ij-rigid in X_α for each $\alpha \in \Lambda$, where $\zeta_i = \pi\zeta_i^\alpha$, for i= 1,2.

Proof The necessity of the condition follows from Theorem 15. For the sufficiency let ϑ be a base for an ultrafilter on $\underset{\alpha}{\pi} X_\alpha$ satisfying $B \cap \zeta_j$-clW $\neq \phi$ for all B $\in \vartheta$

and for all ζ_i-open set W containing $\underset{\alpha}{\pi} A_\alpha$. Then for any $\alpha \in \Lambda$, $p_\alpha(\vartheta)$ is a base for an ultrafilter on X_α, where p_α is the αth projection map. If V is a ζ_i^α-open set containing A_α in X_α, then $p_\alpha^{-1}(V)$ is a ζ_i-open set containing $\underset{\alpha}{\pi} A_\alpha$ and therefore, any $B \in \vartheta$ satisfies $B \cap p_\alpha^{-1}(\zeta_j^\alpha\text{-cl}V) \supset B \cap \zeta_j\text{-cl}p_\alpha^{-1}(V) \neq \phi$. Hence $p_\alpha(B) \cap \zeta_j^\alpha \text{ cl}V \neq \phi$ is satisfied for all $B \in \vartheta$ and for all ζ_i^α-open set V containing A_α. Consequently by Theorem 8 ((e) => (a)) there is an x_α A_α such that $p_\alpha(\vartheta) \xrightarrow{ij-\theta} x_\alpha$. Let $x \in \underset{\alpha}{\pi} X_\alpha$ with $p_\alpha(x) = x_\alpha$ for all $\alpha \in \Lambda$. Then $x \in \underset{\alpha}{\pi} A_\alpha$ and $\vartheta \xrightarrow{ij-\theta} x$ that the proof is complete.

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References

1. R. F. Dickman (Jr.) and J. R. Porter, θ-perfect and θ-absolutely closed functions, Illinois Jour. Math. 21 (1977), 42-60.
2. J. E. Joseph, some remarks on θ-rigidity, Kyungpook Math. Jour. 20 (2) (1980), 245-250.
3. C. G. Kariofillis, On pairwise almost compactness, Ann. Soc. Sci. Bruxelles 100(1986), 129-137.
4. J. C. Kelly, Bitopological space, Proc. London Math. Soc. 13(1963), 71-89.
5. M. N. Mukherjee, On pairwise almost compactness and pairwise H-closedness in bitopological space, Ann. Soc. Sci. Bruxelles T. 96 (1982), 98-106.
6. M. N. Mukherjee, G. K. Banerjee and S. Malakar, Bitopological QHC spaces, Indian J. Pure and Appl. Math. 21(7)(1990), 639-648.
7. S. K. Sen, J. N. Nandi and M. N. Mukherjee, Characterizations of some bitopological separation axioms in terms of ij-θ-closure operator, Mat. Vesnik 44 (1992), 75-82.
8. A. R. Singal and S. P. Arya, On pairwise almost regular space, Glasnik Mat. 6(26)(1971), 335-343.

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IRREDUCIBILITY OF BANACH OPERATOR ALGEBRA IN THE INDUCTION OF NORMS IN THE LINEAR SPACES

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Abstract : Let V and V' be linear spaces over F. Let $L(V,V')$ and $L'(V',V)$ denote the space of linear transformations from V to V' and V' to V respectively, Let (L,L') denote the Γ-algebra of L and L'. Let L and L' be given norms such that (L,L') becomes a Γ-Banach algebra over F. Let (β,β') be an irreducible Γ-subalgebra of the Γ-Banach algebra (L,L'). In this paper, it has been first shown that it is possible to induce norms in V and V' with the help of norms of (L,L') in such a way that V and V' become complete with respect to these norms. Lastly, it has been shown that the norm induced on V is such that every complete (β,β') admissible norm on V is equivalent to this norm.

Introduction : It is known that the norm of an element $f \in B(V)$ [the set of all bounded linear transformations from a linear space V to itself] is defined with the help of norms of elements of V. The converse problem has been answered by C.E.Rickart in the following way : Let $E(V)$ be the set of all linear transformations from V to V and let some norm be given on the algebra $E(V)$ to make it a Banach algebra over F. Let $\bar{\beta}(V)$ be an irreducible subalgebra of $E(V)$. Then norm of V can be induced from that of $E(V)$. For transformations between two normed linear spaces V and V' over F, it is known that norm of an element of $B(V,V')$ [the set of all bounded linear transformations from V to V'] and that of an element of $B'(V',V)$ [similarly defined] can be defined with the help of the norms of both V and V'. This paper, in fact, answers in the affermative, the converse problem regarding the induction of norms of V and V' from that of an irreducible algebra of (L,L').

1. Some known Definitions and Results : [1], [2], [3].

Definition 1. Let V and Γ be two linear spaces over a field F. V is said to be a Γ-algebra over F, if for $a,b,c \in V$; $x,y \in \Gamma$, $\alpha \in F$

(i) $axb \in V$ (ii) $(axb)yc = ax(byc)$ (iii) $\alpha(axb) = \alpha a(xb) = a(\alpha x)b = ax(\alpha b)$ (iv) $ax(b+c) = axb + axc$; $a(x+y)b = axb + ayb$; $(a+b)xc = axc + bxc$.

This Γ-algebra is denoted by (V, Γ).

If V and Γ are normed linear spaces over F, then Γ-algebra (V, Γ) is called a Γ-normed algebra if in addition to (i) – (iv) the condition (v) $\|a x b\| \leq \|a\| \, \|x\| \, \|b\|$ holds.

A Γ-normed algebra $(V,. \Gamma)$ is called a Γ-Banach algebra if V is a Banach space. If Γ is replaced by a linear space V', then also (V,V') is called a Γ-algebra/ Γ.normed algebra / Γ-Banach algebra, as the case may be.

Definition 2. Let (V,V') be a Γ-algebra over F and let $V_1 \subseteq V$, $V_1' \subseteq V'$ be such that $V_1 V_1' V_1 \subseteq V_1$ and (V_1, V_1') is a Γ-algebra with respect to the same compositions as defined in (V,V'), then (V_1,V_1') is called a Γ-subalgebra of the Γ-algebra (V,V').

Definition 3. Let $L(V,V')$, $L'(V',V)$ denote respectively the sapces of all linear maps from V to V' and V' to V ; V,V' being linear spaces over F. Then L is a Γ-algebra over F, denoted by (L,L'), if for $A,B \in L$; $X \in L'$, $x \in V$; $\alpha,\beta \in F$, $(\alpha A + \beta B)x = \alpha A(x) + \beta B(x)$ and $(AXB)x = A[X\,(B(x))]$.

If $B(V,V')$, $B'(V',V)$ denote respectively the spaces of all bounded linear maps from V to V' and V' to V, then $\| A x B \| \leq \|A\| \, \|X\| . \|B\|$ holds and hence (B,B') becomes a Γ-normed algebra under the above compositions. If, further, V' is a Banach space, then (B,B') becomes a Γ-Banach algebra.

Definition 4. Let (β,β') be a Γ -subalgebra of the Γ -algebra (L,L'). A subset $M \subseteq V$ is said to be Γ-invariant with respect to (β,β') if $(f'(f(M)) \subseteq M$, $\forall f \in \beta$, $\forall f' \in \beta'$.

Definition 5. A Γ-subalgebra (β,β') of the Γ-algebra (L,L') is said to be irreducible provided (o) and V are the only Γ-invariant subspaces of V with respect to (β,β').

Definition 6. Let (V,V') be a Γ-Banach algebra. An element $a \in V$ is said to be left a'-quasiinvertible, $a' \in V'$, if there exists a unique $b \in V$ such that $a+b-ba'a = 0 \in V$. Similarly, a is called right a'-quasiinvertible, if $a+b-aa'b = 0$; a is a'-quasiinvertible if it is both left and right a'-quasiinvertible. a is quasiinvertible if it is a'-quasiinvertible for all $a' \in V'$.

The definition of quasiinvertible elements of a Γ-Banach algebra is parallel to that of quasiregular elements of a Γ-ring [4].

Definition 7. Let (V,V') be a Γ-Banach algebra. A subset I of V is called a left Γ-ideal of (V,V') if it is a linear subspace of V such that $VV'I \downarrow I$. It is a right Γ-ideal of (V,V') if $IV'V \subseteq I$. I is called a two sided Γ ideal or simply a Γ-ideal if it is both a left and a right Γ-ideal.

Maximal and minimal Γ-ideals are defined in the usual way.

Definition 8. A left Γ -ideal I of (V,V') is said to be α-modular if there exists $e \varepsilon V$ such that $V - V\alpha e \subseteq I$, for some $\alpha \varepsilon V'$.

I is called modular if $V - VV'e \subseteq I$. Right α-modular /right modular Γ-ideals are defined in a similar manner. A two sided Γ-ideal I is said to be α-modular/modular if it is both left and right α-modular/modular.

Definition 8. Let $\|x\|$ and $|x|$ denote two norms on a linear space V. $\|x\|$ is said to majorize $|x|$, if there exists a constant β such that $|x| \leq \beta \|x\|$.

Definition 9. Let V and V′ be two linear spaces over F. Let (β,β') be a Γ-subalgebra of (L,L'). Let L,L' be given some norms with respect to which (L,L') becomes a Γ-normed algebra. A pair of norms $(|x|, |x_1|^0)$ is called a (β,β') admissible norm on (V,V'); $[|x|, |x_1|^0$ are called (β,β') admissible norms on V and V' respectively] if each operator in $\beta(V,V')$ is bounded with respect to $(|x|, |x_1|^0)$, $x \varepsilon V$, $x_1 \varepsilon V'$ and if $|T|$ majorizes $|T|'$ where $|T|$ and $|T|'$ denotes respectively the norms of $T \varepsilon \beta$ as given be norm of $L(V,V')$ and as induced by $(|x|, |x|^0)$.

Result 1. Every proper α-modular Γ-ideal is contained in a maximal Γ-ideal of the same type.

Result 2. Let (V,V') be a Γ-Banach algebra, then each $x \varepsilon V$ is y'-quasiinvertible if $\|x\| < 1$, where $y' \varepsilon V'$, $\|y'\|$ 1.

Result 3. Let (V,V') be a Γ-Banach algebra. Let L be a proper modular Γ-ideal of (V,V') such that $V - VV'e \subseteq L, e \varepsilon V$, then there is no $x \varepsilon L$ such that $\|e - x\| < 1$.

Corollary : If (V,V') be a Γ-Banach algebra, then the closure of every proper α-modular/modular Γ-ideal is proper, so that maximal α-modular/modular Γ-ideals are closed.

Result 4. Let $E(V) = A(V,V)$ be the set of all linear maps from V to itself . Let for $x \in V$, $\overline{T} \in E(V)$, $\overline{T}(x) = T_1(T(x)$, for some $T \in L(V,V')$, $T_1 \in L'(V',V)$. Then $E(V)$ is an algebra over F, where for $\alpha,\beta \varepsilon F$, $x \in V$, $\overline{T}, \overline{S} \in E(V)$, $(\alpha\overline{T} + \beta\overline{S}) = \alpha\overline{T}(x) + \beta\overline{S}(x)$, $(\overline{T}\,\overline{S})x = \overline{T}(\overline{S}(x))$.

If V,V' are normed linear spaces over F, then $E(V)$ is a normed algebra over F. Further $\overline{T} = T_1T$ satisfies an additional property $|\overline{T}|' \leq \|T_1\| \; |T|$, where $|T|'$, $|T|$, $\|T_1\|$ denote norms of $\overline{T}$, T and T_1.

2. Main Theorems.

Theorem 1. Let $L(V,V')$ and $L(V',V)$ be defined as above. Let L,L' be given suitable norms with respect to which (L,L′) and E(V) becomes respectively a G -Banach algebra and a Γ-Banach algebra over F. Let (β,β') and $\bar{\beta}$ (V) be irreducible Γ-Banach subalgebra and Banach subalgebra of (L,L′) and E(V) respectively. Let u be a fixed element of V and let for each x_1 e V′, x e V,

$$|x_1|^0 = .\inf_{T} |T| \text{ such that } Tu = x_1\ ,\ T\ \varepsilon\ L(V,V'),$$

$$|x| = .\inf_{T_1,T} \|T_1\|\ |T| \text{ such that } T_1T(u) = x,\ T_1\ \varepsilon\ L'(V',V),$$

where |x| , $|x_1|^0$, |T| $\|T_1\|$ denotes norms of x,x_1, T,T_1 respectively. Then V is a Banach space with |x| as the norm; V' is a Banach space with $|x_1|^0$ as the norm. Further, each T ε β is bounded relative to (|x|, $|x_1|^0$) and that $|\bar{T}|$ majorizes $|\bar{T}|'$ where $|\bar{T}|'$ is the norm of T ε β induced by (|x|, $|x_1|^0$). Also |T|′ is majorized by |T|′ where |T|′ is the norm of T as induced by (|x|, $|x_1|^0$) and $|\bar{T}|$ is the norm of T as given by E(V).

Proof : Let £ denote the set of all T in b such that Tu = 0 ε V' Obviously £ is a left Γ-ideal of (β,β'). Now, (β,β') being irreducible, there exists some T ε β, some T_1 εβ' such that $T_1T(u)$ = u. Hence, for arbitary s ε β′ $(ST_1T - S)u = S(T_1(T(u))) - SU$ = SU - SU = 0 e V'. So, $ST_1T - S$ ε δ, ∀ S εβ. Therefore, $\bar{\delta}$ is a α-modular Γ-ideal [α = T_1]. Next, let u be any vector of b not belonging to £. Then U(u) ≠ 0 ε V'. Now, let T ε $\bar{£}$, then for any s ε £, any T_1εβ′,$ST_1U(u)$ = T(u). If T ε β does not belong to $\bar{£}$, then there exists some S eb but not belonging to α and some T_1 ε β′ such that $ST_1U(u)$ = T(u). Hence, in any case, $ST_1U - T$ ε β. Again, for any T ε £, T = ST_1U + (T - ST_1U), where ST_1U - T ε £ (1)

Let us now make a special choice of T viz. T^0 where $T_1T^0(u)$ = u, for some T_1 ∈β'. Obviously, as above, it may be shown that ST_1T^0 - S ε $\bar{£}$. Hence, T cannot belong to £, as £ is a proper α-modular Γ-ideal. So, replacing T of (1) by T^0, it is seen that ST_1U $\notin$ £. This means that £ is maximal. As $\bar{£}$ is a maximal α-modular Γ-ideal of (β,β'), so it follows that $\bar{£}$ is closed. Hence, β/$\bar{£}$ is a Banach space. Let us now consider a map f : β/$\bar{£}$ → V' defined by f : (T)' → T(u). So, it is a linear isomorphism between β/α and V'. Therefore, $|x_1|^0$ (x_1 ε V') is precisely the norm of (T)' ε β/£, (T)' = T + £), transferred to x_1 via this isomorphism. Hence, V' is a Banach space, where $|x|^0$ denotes the norm of x_1. Next, let $\bar{T}$ =T_1T where $\bar{T}$ ε E(V), T εL(V,V'), T_1 e L'(V'V). Let £ = {$\bar{T}$: $\bar{T}$ (u) = T_1 (T(u) = O e V}. Then $\bar{£}$ is a modular ideal of E(V)and it is also closed. It is seen that |x| is precisely the norm of $\bar{\beta}$(V)/£, transferred to x via the isomorphism $(\bar{T})'$ → $T_1T(u)$, where (T)' = T + £. Hence V is a Banach space as β(V)/£ is so. Now we show that |T| majorizes |T|'. In fact, for any x e V,

$$|T(x)|^0 = \inf_{\substack{S\\ Su=Tx}} |S| \le \inf_{\substack{T_1,S\\ T_1^1Su=x}} |TT_1S| \le \inf_{\substack{T_1,S\\ T_1^1Su=x}} |T|\,\|T_1\|\,|S| = |T| \inf_{\substack{T_1,S\\ T_1^1Su=x}} \|T_1\|\,|S| = |T||x|$$

So $|T|' = \inf |T| \le |T|'$. Thus $T \varepsilon \beta(V,V')$ is bounded relative xto $(|x|, |x_1|^0)$ and that $|T|$ majorizes $|T|'$. Similarly, $\overline{T} \varepsilon \overline{\beta}(V)$ is also bounded relative to $(|x|, |x_1|^0)$ and that $|\overline{T}|$ majorizes $|T|''$. This completes the proof.

Theorem 2. Let (β,β') and $|x|$ be defined as in theorem 1. Then every (β, β') admissible norm in V is majorized by $|x|$. Further, every complete (β,β') admissible norm in V is equivalent to $|x|$.

Proof : Let $|x|'$ be any (β,β') admissible norm of x in V. Let for some $T \varepsilon \beta$, some $T_1 \varepsilon \beta', T_1T(u) = x$, then $|x|' = |T_1T(u)|' \le |T_1T|'' \, |u|'$, where $|T_1T|''$ denotes the norm of T_1T as induced by the norm $|x|$ of $x \varepsilon V$ and the norm $|x|^0$ of $x_1 \varepsilon V'$. Also $|T_1T|'$ is majorized by $|T_1T|$. Hence, $|T_1T|' \le \beta|T_1T|$. Thus, $|x|' \le \beta\|T_1\| \, |T| \, |u|'$. So, $\inf_{T_1,T} |x|' \le \beta|u|'$. $\inf_{T_1,T} \|T_1\|' \, |T| = \beta \, |u|' \, |x|$. Hence $|x|' \le \beta \, |u|'|x|$. Thus $|x|$ majorizes $|x|'$. If further, $|x|'$ is also complete, then by consequence of closed graph theorem, it follows that $|x|$ and $|x|'$ are equivalent. This completes the proof.

References

1. Bhattacharya, D.K. and Maity, A.K. :	Regular representation of a Γ-Banach algebra-Proceedings of the Levy seminar in mathematics (1989), Dept. of Pure Mathematics, Cal. University.
2. Maity, A. K. :	Irreducible representation of a Γ-Banach algebra-Jour. Pure Math. Cal. University, Vol. 7, 1990, pp.9-19.
3. Maity, A. K. :	On representation of a Γ-Banach algebra - Ganita' (Lucknow University), Vol. 40. No.1.1989,7-19.
4. Ravisankar T. S. and Sukla U. S. :	Structure of Γ-rings, Pacific Jour.Math.Vol.80,2,1979.
5. Rickart C. E. :	General theory of Banach algebras -D. Van Nostrand Company, Inc. Princetion, New Jersey, 1960.

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